Published by Institute of Physics Publishing for SISSA
Received: *May 5, 2005*
Revised: *August 31, 2005*
Accepted: *September 3, 2005*
Published: *September 29, 2005*

Construction of one-loop $\mathcal{N}=4$ SYM effective action in the harmonic superspace approach

Ioseph L. Buchbinder

Department of Applied Mathematics and Theoretical Physics
University of Cambridge, Centre for Mathematical Sciences
Wilberforce Road, Cambridge, CB3 0WA, U.K., and
Department of Theoretical Physics, Tomsk State Pedagogical University
Tomsk 634041, Russia
E-mail: joseph@tspu.edu.ru

Nikolai G. Pletnev

Department of Theoretical Physics, Institute of Mathematics
Novosibirsk 630090, Russia
E-mail: pletnev@math.nsc.ru

Abstract: We develop a systematic approach to construct the one-loop $\mathcal{N}=4$ SYM effective action depending on both $\mathcal{N}=2$ vector multiplet and hypermultiplet background fields. Beginning with the formulation of $\mathcal{N}=4$ SYM theory in terms of $\mathcal{N}=2$ harmonic superfields, we construct the one-loop effective action using the covariant $\mathcal{N}=2$ harmonic supergraphs and calculate it in $\mathcal{N}=2$ harmonic superfield form for constant abelian strength F_{mn} and corresponding constant hypermultiplet fields. The hypermultiplet-dependent effective action is derived and given by integral over the analytic subspace of harmonic superspace. We show that each term in the Schwinger-De Witt expansion of the low-energy effective action is written as integral over full $\mathcal{N}=2$ superspace.

Keywords: Superspaces, Extended Supersymmetry, Supersymmetric Effective Theories.

Contents

1. Introduction

$\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory attracts a lot of attention due to its unique properties in the quantum domain such as finiteness and superconformal invariance and the remarkable links with string/brane theory (see e.g. the [1, 2] for review). Discovery of AdS/CFT correspondence stimulated a new significant interest to study the various aspects of $\mathcal{N} = 4$ SYM theory. AdS/CFT correspondence [3] states a duality between IIB superstring theory compactified on $AdS_5 \times S^5$ and four-dimensional $\mathcal{N} = 4$ SYM theory in 't Hooft limit. It turnes out, that the low-energy properties of bulk string theory are related to $\mathcal{N} = 4$ supersymmetric gauge quantum field theory. To be more precise, in the limit under consideration, the bulk theory reduces to the higher dimensional classical supergravity encoded by correlation functions of gauge invariant composite operators in $D = 4$, $\mathcal{N} = 4$ SYM theory. Another implementation of $\mathcal{N} = 4$ SYM model to string theory is related to the conjecture that $D3$-brane interactions in static limit are completely described in terms of low-energy $D = 4$, $\mathcal{N} = 4$ SYM effective action on the Coulomb branch [4 − 6]. All this allows us to treat $\mathcal{N} = 4$ SYM quantum field model as an element of superstring theory.

Formulation of $\mathcal{N}=4$ SYM theory possessing manifest off-shell $\mathcal{N}=4$ supersymmetry is unknown so far. Superfield description of the $D=4, \mathcal{N}=4$ SYM theory is realized in terms of scalar superfield $W_{AB}=-W_{BA}, A, B=1,\ldots,4$ which transforms under the six-dimensional representation of the SU(4) internal symmetry group and which is real: $\bar{W}^{AB}=\frac{1}{2}\epsilon^{ABCD}W_{CD}$. This superfield obeys the constrains

$$\mathcal{D}_{\alpha A}W_{BC}=\mathcal{D}_{\alpha[A}W_{BC]}\,, \qquad \bar{\mathcal{D}}^{A}_{\dot\alpha}W_{BC}=-\frac{2}{3}\delta^{A}_{[B}\bar{\mathcal{D}}^{E}_{\dot\alpha}W_{C]E}\,.$$

These constraints imply that the component content of the superfield W_{AB} corresponds to on-shell vector multiplet and includes 6 real scalar fields, 4 Majorana spinor fields and 1 vector field.[1]

From the $\mathcal{N}=2$ supersymmetric point of view, the $\mathcal{N}=4$ vector multiplet consists of the $\mathcal{N}=2$ vector multiplet and the hypermultiplet [8]. Therefore the $\mathcal{N}=4$ SYM action can be treated as some special $\mathcal{N}=2$ supersymmetric theory, with action containing the action of $\mathcal{N}=2$ SYM theory plus the action of a hypermultiplet in adjoint representation coupled to $\mathcal{N}=2$ vector multiplet. In addition, this model possesses the hidden $\mathcal{N}=2$ symmetry and as a result it actually is $\mathcal{N}=4$ supersymmetric. Such a theory is naturally formulated in $\mathcal{N}=2$ harmonic superspace [12, 8]. This formulation simplifies a quantum consideration due to manifest $\mathcal{N}=2$ supersymmetry. Analysis of effective action is more simplified with help of harmonic superfield background field method [14, 15]. Straightforward calculation of higher-loop contributions to effective action in closed form is a very complicated technical problem. We point out that recently an essential progress in development of multi-loop calculations techniques in the harmonic superspace was achieved for $\mathcal{N}=2$ vector multiplet background [16]. Therefore, a study of general structure of possible higher order corrections to the effective action would be very useful (see e.g. [17]).

In this paper the $\mathcal{N}=4$ SYM theory is treated as some special case of generic $\mathcal{N}=2$ gauge models. All such models are formulated in $\mathcal{N}=2$ harmonic superspace in manifest $\mathcal{N}=2$ supersymmetric form. $\mathcal{N}=4$ SYM theory is selected up to an extra hidden on-shell $\mathcal{N}=2$ supersymmetry. Formulation in terms of $\mathcal{N}=2$ superspace allows us to use the known classification of ground state structure in $\mathcal{N}=2$ gauge models (see e.g. [18]). According to such a classification, the branch of the theory possessing the non-zero vacuum expectation values both for scalar from N=2 vector multiplet and for hypermultiplet scalar is called the mixed branch. This just corresponds to the problem under consideration. It is clear, to keep the hidden $\mathcal{N}=2$ supersymmetry of the theory in ground state we have to preserve non-vanishing expectation values not only for $\mathcal{N}=2$ vector multiplet but also for hypermultiplet. The $\mathcal{N}=2$ hidden supersymmetry transformation of values in ground state and their applications were given in [27].[2]

[1]The same multiplet can be described off shell in $\mathcal{N}=3$ harmonic superspace [7, 8]. Quantum aspects of $\mathcal{N}=3$ SYM theory have been discussed in [9]. Structure of effective action of such a theory was studied in [11, 10].

[2]We would like to pay attention that there is no generally accepted terminology in literature concerning the ground state of $\mathcal{N}=4$ SYM theory. One possibilities is to use the terminology from $\mathcal{N}=2$ gauge theories since $\mathcal{N}=4$ SYM is a partial case of these theories. There we follow namely such a possibility

JHEP09(2005)073

At present, it is well known that the exact low-energy quantum dynamics of $\mathcal{N}=4$ SYM theory in $\mathcal{N}=2$ vector multiplet sector for SU(N) gauge group spontaneously broken down to its maximal torus U$(1)^{N-1}$ is described by the non-holomorphic effective potential $\mathcal{H}(\mathcal{W},\bar{\mathcal{W}})$, depending on $\mathcal{N}=2$ strengths $\mathcal{W},\bar{\mathcal{W}}$ [19, 20, 15]. We emphasize that the structure of $\mathcal{H}(\mathcal{W},\bar{\mathcal{W}})$ is so unique that it can be obtained entirely on the symmetry grounds of scale independence and R-invariance up to a numerical factor [19, 21]. Moreover, the potential $\mathcal{H}(\mathcal{W},\bar{\mathcal{W}})$ gets neither perturbative quantum corrections beyond one-loop nor instanton corrections [19], (see also discussion of non-holomorphic potential in $\mathcal{N}=2$ SYM theories [21, 22] and structure of next-to-leading two-loop contributions to effective action [23, 6, 16]). All these properties are very important for understanding of the low-energy quantum dynamics in $\mathcal{N}=4$ SYM theory and its applications. In particular, the effective potential $\mathcal{H}(\mathcal{W},\bar{\mathcal{W}})$ describes the leading terms in the interaction between parallel D3-branes in superstring theory.

To clarify the restrictions on the effective action, stipulated by $\mathcal{N}=4$ supersymmetry, to describe complete structure of the effective action and to gain a deeper understanding of the $\mathcal{N}=4$ SYM/supergravity correspondence, we have to find an effective action not only in $\mathcal{N}=2$ vector multiplet sector but its dependence on all the fields of $\mathcal{N}=4$ vector multiplet. The problem of the $\mathcal{N}=4$ SYM effective action on the so called mixed branch remained unstudied for a long time. Recently, the complete exact low-energy effective potential $\mathcal{L}_q(X)$ ($X=-\frac{q^{ia}q_{ia}}{\mathcal{W}\bar{\mathcal{W}}}$) containing the dependence both on $\mathcal{N}=2$ gauge superfields and hypermultiplets has been obtained [24]. It has been shown that the algebraic restrictions imposed by the hidden $\mathcal{N}=2$ supersymmetry on a structure of the low-energy effective action in $\mathcal{N}=2$ harmonic superspace approach turns out to be so strong that they allow us to restore the dependence of the low-energy effective action on the hypermultiplets on basis of the known non-holomorphic effective potential $\mathcal{H}(\mathcal{W},\bar{\mathcal{W}})$. As a result, the additional hypermultiplet-dependent contribution to low-energy effective action containing the on-shell $\mathcal{W},\bar{\mathcal{W}}$ and the hypermultiplet q^{ia} superfields has been obtained. The leading low-energy effective lagrangian $\mathcal{L}_q(X)$ was found in ref. [24] on a purely algebraic ground. Then, it was shown in paper [25], that the effective lagrangian $\mathcal{L}_q(X)$ can be derived using the harmonic supergraph techniques and harmonic superspace background field method. Later, the structure of the one-loop effective action beyond leading approximation has been found in [27] on the base of formulation of $\mathcal{N}=4$ SYM theory in terms of $\mathcal{N}=1$ superfields and exploring the derivative expansion techniques in $\mathcal{N}=1$ superspace [28]. Although such a formulation preserves a fewer number of manifest supersymmetries than in the harmonic superspace approach, the supersymmetric R_ξ-gauges and a special prescription on restoration of the $\mathcal{N}=2$ supersymmetric form make possible to construct the effective action including dependence on arbitrary powers of the abelian strength F_{mn} and special R-symmetry invariant combination of the constant component fields ϕ, f^{ia} of hypermultiplet.

The present paper is devoted to analysis of hypermultiplet dependence of $\mathcal{N}=4$ SYM low-energy effective action in $\mathcal{N}=2$ harmonic superspace. We compute the low-energy effective action for the space-time independent $\mathcal{N}=2$ vector multiplet and hypermultiplet background under the followig assumptions: (i) $\mathcal{W}|_{\theta=0} = \text{const}, D^{\pm}_{\alpha}\mathcal{W}|_{\theta=0} = \text{const}, D^{-}_{\alpha}D^{+}_{\beta}\mathcal{W}|_{\theta=0} = \text{const}$, (ii) the background hypermultiplet is on-shell, $q^{\pm a} = q^{ia}u^{\pm}_{i}$ where q^{ia} does not depend on harmonics [24] and is considered constant. This means that, in this paper, we treat the effective action in the hypermultiplet sector as a series in spinor derivatives of q^{+a} and study the simplest approximation corresponding to contributions to the effective action which do not depend on any derivatives of q^{+a}. Comparing with paper [25], we consider the effective action beyond the leading low-energy approximation in $\mathcal{N} = 2$ vector multiplet sector and take into account all powers of the abelian strength F_{mn}. On the other hand, comparing with paper [27], we work completely in terms of $\mathcal{N} = 2$ harmonic superspace at all steps of consideration and justify the special heuristic prescription concerning a restoration of manifest $\mathcal{N} = 2$ supersymmetric form of effective action, which has been used in [27]. As a result we obtain a proper-time representation of low-energy effective action written as integral over the analytic subspace of harmonic superspace.

This paper is organized as follows. In section 2, we discuss the construction of $\mathcal{N} = 4$ SYM model in $\mathcal{N} = 2$ harmonic superspace and structure of perturbation theory in this model. Section 3 is devoted to a generic procedure of finding the one-loop effective action in the hypermultiplet sector. In section 4, we show how one can sum up an infinite series of covariant harmonic supergraphs with arbitrary number of hypermultiplet legs on non-trivial $\mathcal{N} = 2$ vector multiplet background to get a form of the effective action which can be studied on the base of proper-time method. Section 5 is devoted to calculating the one-loop effective action with help of symbol operator techniques we develop in $\mathcal{N} = 2$ harmonic superspace. In section 6, we derive the final result for the one-loop effective action in the form of an integral over the analytic subspace of harmonic superspace. We also show that this result leads to spinor covariant derivative expansion of the effective action and find two first terms of this expansion in explicit form. We demonstrate that each term of the expansion can be rewritten as integral over full $\mathcal{N}$=2 superspace. In section 7, we summarize our main results.

2. $\mathcal{N} = 4$ SYM theory in $\mathcal{N} = 2$ harmonic superspace

The harmonic superspace is an universal construction for formulating the arbitrary $\mathcal{N} = 2$ supersymmetric theories in a manifestly $\mathcal{N} = 2$ supersymmetric way. $\mathcal{N} = 2$ harmonic superspace was introduced in Ref. [12] to construct off-shell $\mathcal{N} = 2$ multiplets and develop a superfield formulation of $\mathcal{N} = 2$ models in terms of unconstrained superfields. This approach extends the conventional $\mathcal{N} = 2$ superspace $z = (x^m, \theta^{\alpha}_i, \bar{\theta}^{\dot{\alpha}i})$ by the two-sphere $S^2 = \mathrm{SU}(2)/\mathrm{U}(1)$ parameterized by harmonics u^{+i} and their conjugate u^-_i. Throughout this paper we follow the conventions of the book [8]. Harmonic superspace with coordinates z, u contains the analytic subspace parameterized by the variables $\zeta = (x^m_A, \theta^{+\alpha}, \bar{\theta}^{+}_{\dot{\alpha}}, u^+_i, u^-_i)$, where the analytic basis is defined by

$$x^m_A = x^m - 2i\theta^{(i}\sigma^m\bar{\theta}^{j)}u^+_i u^-_j\,, \theta^{\pm}_{\alpha} = u^{\pm}_i\theta^i_{\alpha}, \bar{\theta}^{\pm}_{\dot{\alpha}} = u^{\pm}_i\bar{\theta}^i_{\dot{\alpha}}\,.$$

The spinor covariant derivatives in the central and the analytic bases (see discussion of these bases in [8]) are related by $D^{\pm}_{\alpha} = u^{\pm}_{i} D^{i}_{\alpha}, \bar{D}^{\pm}_{\dot{\alpha}} = u^{\pm}_{i} D^{i}_{\dot{\alpha}}$. A very important feature is that the operators $D^{+}_{\alpha}, \bar{D}^{+}_{\dot{\alpha}}$ strictly anticommuting. A covariantly analytic superfield $\Phi^{(p)}(z,u)$[3] is defined to be annihilated by these operators, $D^{+}_{\alpha}\Phi^{(p)} = \bar{D}^{+}_{\dot{\alpha}}\Phi^{(p)} = 0$.

The two basic $\mathcal{N}=2$ ingredients of the $\mathcal{N}=4$ SYM theory are a hypermultiplet and a $\mathcal{N}=2$ vector multiplet. In harmonic superspace hypermultiplets are described by an analytic superfield of U(1) charge equal to $+1$,

$$D^{+}_{\alpha} q^{+} = \bar{D}^{+}_{\dot{\alpha}} q^{+} = 0 \Rightarrow q^{+} = q^{+}(x_A, \theta^{+}, \bar{\theta}^{+}, u)\,.$$

In the on-shell case the hypermultiplet satisfies the condition of harmonic analyticity $D^{++}q^{+} = 0$ and forms the so called an ultrashort superfield. Here

$$D^{++} = u^{+i}\frac{\partial}{\partial u^{-i}} - 2i(\theta^{+}\sigma^{m}\bar{\theta}^{+})\frac{\partial}{\partial x^{m}_{A}}$$

is the covariant harmonic derivative in the analytic basis, which also plays the role of raising operator in the $su(2)$ algebra: $[D^{++}, D^{--}] = D^{0}$, $[D^{0}, D^{\pm\pm}] = \pm 2D^{\pm\pm}$, $[D^{++}, D^{-}] = D^{+}$. The classical action of the hypermultiplet is:

$$S_{\rm hyper} = -\int d\zeta^{(-4)} du \breve{q}^{+} D^{++} q^{+}\,, \tag{2.1}$$

where the integration is carried out over the analytic subspace. The rules of harmonic integration are given in [8].

The $\mathcal{N}=2$ vector multiplet is described by a real analytic superfield $V^{++}(x,\theta^{+},\bar{\theta}^{+},u)$ with U(1) charge equal to $+2$. It plays the role of the gauge connection in the covariantized harmonic derivative

$$\mathcal{D}^{++} = D^{++} + igV^{++}\,,$$

so that the "flat" commutation relations with $D^{+}_{(\alpha,\dot{\alpha})}$ are preserved: $[\mathcal{D}^{++}, \mathcal{D}^{+}_{\alpha,\dot{\alpha}}] = 0$. In the Wess-Zumino gauge the component contents of V^{++} is reduced to the off-shell $\mathcal{N}=2$ vector multiplet. Namely this superfield V^{++} is an unconstrained prepotential for $\mathcal{N}=2$ SYM theory and all other objects, for example the superfield strength $\mathcal{W}$, are expressed in its terms [12, 8]. The superfield strength is expressed through non analytic superfield V^{--} satisfying the equation $D^{++}V^{--} - D^{--}V^{++} + i[V^{++}, V^{--}] = 0$. This equation has the solution in form of the power series

$$V^{--} = \sum_{n=1}^{\infty}\int d^{12}z du_1\ldots du_n(-i)^{n+1}\frac{V^{++}(z,u_1)\ldots V^{++}(z,u_n)}{(u^{+}u^{+}_1)(u^{+}_1u^{+}_2)\ldots(u^{+}_n u^{+})}\,, \tag{2.2}$$

which is a non-local functional of V^{++} in harmonic sector. The conditions of u independence of $\mathcal{W}$, $\mathcal{D}^{\pm\pm}\mathcal{W} = 0$ lead to

$$\mathcal{W} = -\frac{1}{4}(\bar{\mathcal{D}}^{+})^{2}V^{--}, \bar{\mathcal{W}} = -\frac{1}{4}(\mathcal{D}^{+})^{2}V^{--}\,. \tag{2.3}$$

[3]Here the superscript p refers to the harmonic U(1) charge, $D^{0}\Phi^{(p)} = p\Phi^{(p)}$

The remaining properties of $\mathcal{W}, \bar{\mathcal{W}}$ are covariant chirality (antichirality), $\mathcal{D}^{\pm}_{\dot\alpha}\bar{\mathcal{W}} = \bar{\mathcal{D}}^{\pm}_{\dot\alpha}\mathcal{W} = 0$, and the Bianchi identity: $\mathcal{D}^{\alpha i}\mathcal{D}^{j}_{\alpha}\mathcal{W} = \bar{\mathcal{D}}^{i}_{\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha j}\bar{\mathcal{W}}$. For further use we will write down also the algebra of covariant derivatives:

$$
\begin{aligned}
\{\mathcal{D}^+_\alpha, \mathcal{D}^+_\beta\} &= \{\bar{\mathcal{D}}^+_{\dot\alpha}, \bar{\mathcal{D}}^+_{\dot\beta}\} = \{\mathcal{D}^+_\alpha, \bar{\mathcal{D}}^+_{\dot\alpha}\} = 0\,,\\
\{\mathcal{D}^+_\alpha, \mathcal{D}^-_\beta\} &= -2i\varepsilon_{\alpha\beta}\bar{\mathcal{W}}\,, \qquad \{\bar{\mathcal{D}}^+_{\dot\alpha}, \bar{\mathcal{D}}^-_{\dot\beta}\} = 2i\varepsilon_{\dot\alpha\dot\beta}\mathcal{W}\,,\\
\{\bar{\mathcal{D}}^+_{\dot\alpha}, \mathcal{D}^-_\alpha\} &= -\{\mathcal{D}^+_\alpha, \bar{\mathcal{D}}^-_{\dot\alpha}\} = 2i\mathcal{D}_{\alpha\dot\alpha}\,,\\
[\mathcal{D}^{++}, \mathcal{D}^-_\alpha] &= \mathcal{D}^+_\alpha\,, \qquad [\mathcal{D}^{++}, \bar{\mathcal{D}}^-_{\dot\alpha}] = \mathcal{D}^+_{\dot\alpha}\,,\\
[\mathcal{D}^\pm_\alpha, \mathcal{D}_{\beta\dot\beta}] &= \varepsilon_{\alpha\beta}\bar{D}^\pm_{\dot\beta}\bar{\mathcal{W}}\,, \qquad [\bar{\mathcal{D}}^\pm_{\dot\alpha}, \mathcal{D}_{\beta\dot\beta}] = \varepsilon_{\dot\alpha\dot\beta}D^\pm_\beta\mathcal{W}\,,\\
[\mathcal{D}_{\alpha\dot\alpha}, \mathcal{D}_{\beta\dot\beta}] &= \frac{1}{2i}\{\varepsilon_{\alpha\beta}\bar{D}^+_{\dot\alpha}\bar{D}^-_{\dot\beta}\bar{\mathcal{W}} + \varepsilon_{\dot\alpha\dot\beta}D^-_\alpha D^+_\beta\mathcal{W}\}\,.
\end{aligned}
\tag{2.4}
$$

Description of the $\mathcal{N}=4$ SYM theory in $\mathcal{N}=2$ harmonic superspace includes hypermultiplet and vector multiplet superfields. The hypermultiplet superfield q^+ belongs to the adjoint representation of the gauge group and is minimally coupled to the vector multiplet. Action of $\mathcal{N}=4$ SYM theory in the approach under consideration has the form:

$$
S = \frac{1}{2g^2}\mathrm{tr}\int d^8z\mathcal{W}^2 - \frac{1}{2}\mathrm{tr}\int d\zeta^{(-4)}q^{+a}(D^{++} + iV^{++})q^+_a\,. \tag{2.5}
$$

Here $a = 1, 2$ is the index of the Pauli-Gürsey rigid SU(2) symmetry: $q^+_a = (q^+, \breve{q}^+), q^{+a} = \varepsilon^{ab}q^+_a = (\breve{q}^+, -q^+)$, and $d^8z = d^4xd^2\theta^+d^2\theta^-du$ is the chiral superspace integration measure. Similarly, $d\zeta^{(-4)} = d^4xd^2\theta^+d^2\bar\theta^+du$ is the analytic measure. The off-shell action (2.5) allows us to develop a manifest $\mathcal{N}=2$ supersymmetric quantization. Moreover, this action is invariant under hidden on-shell extra $\mathcal{N}=2$ supersymmetry transformation [8] realized in terms of the vector and hypermultiplet superfields as follows:

$$
\delta V^{++} = (\varepsilon^{\alpha a}\theta^+_a + \bar\varepsilon^a_{\dot\alpha}\bar\theta^{+\dot\alpha})q^+_a\,, \qquad \delta q^+_a = -\frac{1}{2}(D^+)^4[(\varepsilon^\alpha_a\theta^-_\alpha + \bar\varepsilon_{\dot\alpha a}\bar\theta^{-\dot\alpha})V^{--}]\,. \tag{2.6}
$$

As a result, the model is (on-shell) $\mathcal{N}=4$ supersymmetric.

The on-shell structure of model is defined in terms of solutions to the corresponding equation of motion

$$
D^{++}q^{+a} + ig[V^{++}, q^{+a}] = 0\,, \qquad (\mathcal{D}^+)^2\mathcal{W} = [q^{+a}, q^+_a]\,. \tag{2.7}
$$

The simplest solution to these equations in abelian case forms a set of constant background fields, which transform linearly through each other mixing up the $\mathcal{W}, \bar{\mathcal{W}}$ with q^+_a under hidden $\mathcal{N}=2$ supersymmetry transformation [20, 15]:

$$
\begin{aligned}
\delta\mathcal{W} &= \frac{1}{2}\bar\varepsilon^{\dot\alpha a}\bar{D}^-_{\dot\alpha}q^+_a\,, \quad \delta\bar{\mathcal{W}} = \frac{1}{2}\varepsilon^{\alpha a}D^-_\alpha q^+_a\,,\\
\delta q^+_a &= \frac{1}{4}(\varepsilon^\alpha_a D^+_\alpha\mathcal{W} + \bar\varepsilon^{\dot\alpha}_a\bar{D}^+_{\dot\alpha}\bar{\mathcal{W}})\,.
\end{aligned}
\tag{2.8}
$$

Generic vacuum of any $\mathcal{N}=2$ superconformal models, like for example $\mathcal{N}=4$ SYM theory, includes only massless U(1) vector multiplets and massless neutral hypermultiplets, since charged hypermultiplets get masses by means of the Higgs mechanism. The manifold of vacua is determined by the conditions of vanishing scalar potential (F-flatness plus D-flatness). The set of vacua with only massless neutral hypermultiplets forms the "Higgs branch" of the theory. The set with only U(1) vector multiplets forms the "Coulomb branch". At last, the set of vacua with both kinds of the multiplets forms the "mixed branch". Thus, the low energy fields propagating on the mixed branch are massless neutral scalars, spinors and U(1) vectors which form the on shell superfields $\mathcal{W}, \bar{\mathcal{W}}, q^{+a}, q^{-a}$ possessing the properties

$$
\begin{aligned}
&(D^{\pm})^2\mathcal{W} = (\bar{D}^{\pm})^2\bar{\mathcal{W}} = 0\,,\\
&D^{++}q^{+a} = D^{--}q^{-a} = 0\,, \qquad q^{-a} \equiv D^{--}q^{+a}\,, \qquad D^-_\alpha q^{-a} = \bar{D}^-_{\dot\alpha} q^{-a} = 0\,.
\end{aligned}
$$

Further we will consider the low-energy effective action in $\mathcal{N}=4$ SYM theory just on the mixed branch in the above sense. On the other hand, since the $\mathcal{N}=2$ vector multiplet and the hypermultiplet form the $\mathcal{N}=4$ vector multiplet in the case under consideration, we can treat the vacuum state as an unique Coulomb branch of the model. We want to emphasize that use of the various terms for vacuum state in this theory is a matter of convention.

The manifestly $\mathcal{N}=2$ supersymmetric Feynman rules in harmonic superspace have been developed in [12] (see also [8, 13]). Calculations of quantum corrections may contain potentially dangerous harmonic singularities, that is harmonic distributions at coinciding points. The problem of coinciding harmonic singularities in the framework of harmonic supergraph Feynman rules was first discussed in [29] where some solution to the problem was considered. The background field method for constructing the effective action in harmonic superspace has been developed in refs. [14, 15] (see also [30] for construction of the background field method in standard $N=2$ superspace). This method allows us to find the effective action for arbitrary $\mathcal{N}=2$ supersymmetric gauge model in a form preserving the manifest $\mathcal{N}=2$ supersymmetry and classical gauge invariance in quantum theory. In framework of background field method the fields V^{++}, q^+ are split into classical V^{++}, q^+ and quantum v^{++}, Q^+ fields with imposing the gauge conditions only on quantum fields. The Feynman rules are based on the quantum action $S_{\rm quant}$ of the form $S_{\rm quant} = S_2 + S_{\rm int}$ where S_2 is a quadratic form in quantum fields and ghosts and $S_{\rm int}$ describes interaction. Both action S_2 and action $S_{\rm int}$ depend on background fields. All details are given in [14, 15].

The action S_2 defines the propagators depending on background fields. Further we use the background covariant gauge $\mathcal{D}^{++}v^{++}=0$. In this case the propagator of quantum gauge superfield has the form

$$
G^{(2,2)}(1,2) = \frac{1}{2\widehat{\Box}_1\widehat{\Box}_2}(\mathcal{D}_1^+)^4(\mathcal{D}_2^+)^4\left\{\delta^{12}(z_1-z_2)(D_2^{--})^2\delta^{(-2,2)}(u_1,u_2)\right\}. \tag{2.9}
$$

We emphasize that this propagator is analytic superfield in each argument. The propagator of the Faddeev-Popov ghosts b and c is written as follows

$$
G^{(0,0)}(1,2) = i\left\langle b(1)c^T(2)\right\rangle = \frac{1}{\widehat{\Box}_1}(\mathcal{D}_1^+)^4(\mathcal{D}_2^+)^4\left\{\delta^{12}(z_1-z_2)\frac{(u_1^-u_2^-)}{(u_1^+u_2^+)^3}\right\}. \tag{2.10}
$$

The q^+ hypermultiplet propagator associated with the action (2.5) for external V^{++} has the form

$$G^{(1,1)}(1,2) = i\left\langle q^{+a}(1)\breve{q}^{+}_{b}(2)\right\rangle = -\delta^a_b \frac{1}{\widehat{\Box}_1}(\mathcal{D}^+_1)^4(\mathcal{D}^+_2)^4\left\{\delta^{12}(z_1 - z_2)\frac{1}{(u^+_1 u^+_2)^3}\right\}. \qquad (2.11)$$

The propagators contain the operator $\widehat{\Box} = -\frac{1}{2}(\mathcal{D}^+)^4(\mathcal{D}^{--})^2$ which transforms each covariantly analytic superfield into a covariantly analytic one. On the space of such superfields, the operator $\widehat{\Box}$ is equivalent to the second-order Laplace like differential operator

$$\widehat{\Box} = \frac{1}{2}\mathcal{D}^{\dot\alpha\alpha}\mathcal{D}_{\alpha\dot\alpha} + \frac{i}{2}(\mathcal{D}^{+\alpha}\mathcal{W})\mathcal{D}^-_\alpha + \frac{i}{2}(\bar{\mathcal{D}}^+_{\dot\alpha}\bar{\mathcal{W}})\bar{\mathcal{D}}^{-\dot\alpha} - \frac{i}{4}(\bar{\mathcal{D}}^+_{\dot\alpha}\bar{\mathcal{D}}^{+\dot\alpha}\bar{\mathcal{W}})\mathcal{D}^{--} + \\ + \frac{i}{8}[\mathcal{D}^{+\alpha}, \mathcal{D}^-_\alpha]\mathcal{W} + \frac{1}{2}\{\mathcal{W}, \bar{\mathcal{W}}\}\,. \qquad (2.12)$$

This is a consequence of algebra of covariant derivatives (2.4). It is remarkable that the differential part of $\widehat{\Box}$ is uniquely determined from the requirements that (i) $\widehat{\Box}$ a constructed in terms of the covariant derivatives only; (ii) $\widehat{\Box}$ transforms every covariantly analytic superfield into a covariantly analytic one. This operator is said to be the analytic d'alambertian. Among the important properties of $\widehat{\Box}$ is [14] : $[\mathcal{D}^+_{(\alpha,\dot\alpha)}, \widehat{\Box}] = 0$. The $\mathcal{N} = 2$ propagators have a complicated structure due to nontrivial dependence on harmonics. Fortunately, as is shown recently [31], the harmonic dependence of the $\mathcal{N} = 2$ propagators simplifies drastically if the background vector multiplet satisfies the classical equations of motion $\mathcal{D}^{ij}\mathcal{W} = \bar{\mathcal{D}}^{ij}\bar{\mathcal{W}} = 0$. In this case the harmonic dependence of the propagators is completely factorization what helps to keep the harmonic dependence of $\mathcal{N} = 2$ supergraphs under control.

Evaluation of effective action within the background field method is accompanied often by use of proper time or heat kernel techniques. These techniques allow one to sum up efficiently an infinite set of Feynman diagrams with increasing number of insertions of the background fields and to develop a background field derivative expansion of the effective action in manifestly gauge covariant way. The background field method and heat kernel techniques for $\mathcal{N} = 1$ SYM theories were well-developed (see [32, 33] for reviews). The background field method in harmonic superspace was elaborated in refs. [14], some of its important applications were reviewed in [15]. However, until recently, the many aspects of heat kernel techniques in $\mathcal{N} = 2$ superspace remained almost totally unstudied.[4] In section 5, we use the $\mathcal{N} = 2$ heat kernel techniques to construct the one-loop effective action of $\mathcal{N} = 4$ SYM theory in $\mathcal{N} = 2$ harmonic superspace and extend the one-loop results of [24, 25] for the leading low-energy quantum corrections to the next-to-leading contributions to the effective action depending both on $\mathcal{N} = 2$ vector multiplet and on $\mathcal{N} = 2$ hypermultiplet.

3. One-loop effective action in hypermultiplet sector

We consider the $\mathcal{N} = 4$ SYM theory with gauge group $SU(2)$ formulated in $\mathcal{N} = 2$ harmonic superspace. Action of the model has the form (2.5). On the mixed branch the gauge group $SU(2)$ is broken down to U(1) so that background fields V^{++}, q^+ lie in the

[4] Some new results were presented in [34].

Cartan subalgebra of the gauge group. We start, like in [25], with carrying out background-quantum splitting by the rule $q^{+a} \to q^{+a} + Q^{+a}, \quad V^{++} \to V^{++} + gv^{++}$. Here q^{+a}, V^{++} are the background fields and Q^{+a}, v^{++} are the quantum ones. For one-loop calculations it is sufficient to consider only the part of quantum action S_{quant} which is quadratic in quantum superfields:

$$S_2 = -\frac{1}{2}\text{tr}\int d\zeta^{(-4)}[v^{++}\widehat{\Box}v^{++} + Q^{+a}(D^{++} + iV^{++})Q^+_a + Q^{+a}(igv^{++})q^+_a + \\ + q^{+a}(igv^{++})Q^+_a] + \cdots . \tag{3.1}$$

Where ... means the ghost contribution. The operator $\widehat{\Box}$ includes the background $\mathcal{N}=2$ superfield strengths $\mathcal{W}, \bar{\mathcal{W}}$ which acts in the adjoint representation of the gauge group and has the form (2.12). The set of the background superfields is on-shell, $(\mathcal{D}^+)^2\mathcal{W} = 0, \quad \mathcal{D}^{++}q^{+a} = 0$ and satisfies the relations $\mathcal{D}^{+\alpha}\mathcal{D}^-_\alpha\mathcal{W} = 0$ and $\mathcal{D}^{\pm\alpha}\mathcal{D}^\pm_\alpha\mathcal{W} = 0$. We redefine $g\,Q^+ \to Q^+$ and write the background superfields $V^{++} = \tau_3 V_3^{++}$, $q^+ = \tau_3 q_3^+$ and hence $\mathcal{W} = \tau_3\mathcal{W}_3$. Here $\tau_i = \frac{1}{\sqrt{2}}\sigma_i$ are generators of $su(2)$ algebra:

$$[\tau_i, \tau_j] = i\sqrt{2}\epsilon_{ijk}\tau_k\,, \qquad \text{tr}(\tau_i\tau_j) = \delta_{ij}\,.$$

For the background belonging to an abelian subgroup we have also the further restrictions: $\mathcal{D}^{\pm\alpha}\mathcal{W} = D^{\pm\alpha}\mathcal{W}$, and similarly for $\bar{\mathcal{W}}$ with $D, \bar{D}$ being "flat" derivatives. Then taking into account the on-shell conditions and that all quantum superfields are in the adjoint representation $v^{++} = v_i^{++}\tau_i$, $Q^{+a} = Q_i^{+a}\tau_i$ one gets

$$\widehat{\Box}v^{++} = \Box_{cov}v^{++} + \frac{i}{2}[D^{+\alpha}\mathcal{W}, \mathcal{D}^-_\alpha v^{++}] + \frac{i}{2}[\bar{D}^{+\dot{\alpha}}\bar{\mathcal{W}}, \bar{\mathcal{D}}^-_{\dot{\alpha}}v^{++}] + [\mathcal{W}, [\bar{\mathcal{W}}, v^{++}]]\,. \tag{3.2}$$

Here $\Box_{cov} = \frac{1}{2}\mathcal{D}^{\alpha\dot{\alpha}}\mathcal{D}_{\alpha\dot{\alpha}}$, $q^{+a}v^{++}Q^+_a = q^{+a}[v^{++}, Q^+_a]$. We also have $\mathcal{D}^{++}q^1 = D^{++}q^1 + \sqrt{2}V^{++}q^2$, $\mathcal{D}^{++}q^2 = D^{++}q^2 - \sqrt{2}V^{++}q^1$, $\mathcal{D}^{++}q^3 = D^{++}q^3$, $\mathcal{D}_m = D_m + \sqrt{2}A_m$.

Thus, the quadratic part of the action is written as

$$\begin{aligned} S_2 = -\frac{1}{2}\int d\zeta^{(-4)}\Big\{ & v_1^{++}(\Box_{cov} + 2\mathcal{W}\bar{\mathcal{W}})v_1^{++} + v_2^{++}(\Box_{cov} + 2\mathcal{W}\bar{\mathcal{W}})v_2^{++} + \\ & + \frac{1}{\sqrt{2}}v_1^{++}((D^{+\alpha}\mathcal{W})D^-_\alpha + (\bar{D}^+_{\dot{\alpha}}\bar{\mathcal{W}})\bar{D}^{-\dot{\alpha}})v_2^{++} - \\ & - \frac{1}{\sqrt{2}}v_2^{++}((D^{+\alpha}\mathcal{W})D^-_\alpha + (\bar{D}^+_{\dot{\alpha}}\bar{\mathcal{W}})\bar{D}^{-\dot{\alpha}})v_1^{++} + \\ & + v_3^{++}\Box v_3^{++} + Q_i^{+a}D^{++}Q^+_{ia} + \\ & + Q_1^{+a}(\sqrt{2}V^{++})Q^+_{2a} + Q_2^{+a}(-\sqrt{2}V^{++})Q^+_{1a} + \\ & + v_2^{++}\sqrt{2}(q^{+a}Q^+_{a1} - Q_1^{+a}q^+_a) + v_1^{++}\sqrt{2}(Q_2^{+a}q^+_a - q^{+a}Q^+_{a2})\Big\} + \cdots . \end{aligned} \tag{3.3}$$

Here and further we suppress the index 3 at background superfields, ... means the ghost contribution. It can be seen from (3.3) that only the components of quantum superfields carrying indices 1, 2 have a non-trivial background-dependent propagator, while the quantum superfield components with index 3 do not interact with the background and totally decouple.

JHEP09(2005)073

We define the new complex quantum superfields

$$\chi^{++} = \frac{1}{\sqrt{2}}(v_1^{++} + iv_2^{++}), \qquad \bar{\chi}^{++} = \frac{1}{\sqrt{2}}(v_1^{++} - iv_2^{++}),$$
$$\eta^{+a} = \frac{1}{\sqrt{2}}(Q_1^{+a} + iQ_2^{+a}), \qquad \bar{\eta}^{+a} = \frac{1}{\sqrt{2}}(Q_1^{+a} - iQ_2^{+a}).$$

As a result, the action (3.3) takes the form

$$\begin{aligned} S_2 = -\int d\zeta^{(-4)} \Big\{ & \bar{\chi}^{++}(\Box_{\rm cov} + 2\mathcal{W}\bar{\mathcal{W}} - \frac{i}{\sqrt{2}}((D^{+\alpha}\mathcal{W})D^-_\alpha + (\bar{D}^+_{\dot\alpha}\bar{\mathcal{W}})\bar{D}^{-\dot\alpha}))\chi^{++} + \\ & + \bar{\eta}^{+a}(D^{++} - i\sqrt{2}V^{++})\eta^+_a - i\chi^{++}\sqrt{2}q^{+a}\bar{\eta}^+_a + i\bar{\chi}^{++}\sqrt{2}q^{+a}\eta^+_a + \\ & + \frac{1}{2}v_3^{++}\Box v_3^{++} + \frac{1}{2}Q_3^{+a}D^{++}Q^+_{3a} \Big\}. \end{aligned} \tag{3.4}$$

This form of action is very convenient for perturbative calculations. It has a diagonal part S_0 that defines the propagators and a non-diagonal part V responsible for interaction. The non-interacting fields Q_3^{a+}, v_3^{++} will be omitted. As the next step, we introduce the operator

$$\widehat{\Box}_{\rm short} = \Box_{\rm cov} + 2\mathcal{W}\bar{\mathcal{W}} - \frac{i}{\sqrt{2}}((D^{+\alpha}\mathcal{W})\mathcal{D}^-_\alpha + (\bar{D}^+_{\dot\alpha}\bar{\mathcal{W}})\bar{\mathcal{D}}^{-\dot\alpha}), \tag{3.5}$$

which is obtained from (2.12) by keeping only the superfields $\mathcal{W}, \bar{\mathcal{W}}$ associated with the unbroken U(1) subgroup and putting them on shell. Note that on shell the form of the operator (2.12) and (3.5) is related to each other by $\mathcal{W} \to \frac{\mathcal{W}}{-\sqrt{2}}$, $\bar{\mathcal{W}} \to \frac{\bar{\mathcal{W}}}{-\sqrt{2}}$. Throughout the paper, we will still be using on shell notation (2.12) for (3.5). Now it is easy to read off the Feynman rules for calculating the effective action, which are derived in a similar manner to [12, 8, 14]. For the propagators of the gauge fields χ^{++} and $\bar{\chi}^{++}$ and the hypermultiplets η^+_a and $\bar{\eta}^{+a}$, we use (2.9) and (2.11) respectively. Vertices are taken directly from the V in the form

$$V = -i\chi^{++}\sqrt{2}q^{+a}\bar{\eta}^+_a + i\bar{\chi}^{++}\sqrt{2}q^{+a}\eta^+_a.$$

The Feynman rules look standard. We emphasize only the important point. In each vertex which includes the integral over analytic subspace one can use the factor $(D^+)^4$ from one propagator and convert the integral over $d\zeta^{(-4)}$ to integral over full $\mathcal{N}=2$ measure $d^{12}z$.

We point out that the functional change of variables in (3.4):

$$\chi^{++}(1) \to \chi^{++}(1) - i\int d\zeta_2^{(-4)} G^{(2,2)}(1|2)q^{+a}(2)\eta^+_a(2),$$
$$\bar{\chi}^{++}(1) \to \bar{\chi}^{++}(1) + i\int d\zeta_2^{(-4)} G^{(2,2)}(1|2)q^{+a}(2)\bar{\eta}^+_a(2),$$

with $G^{2,2}(1|2)$ (2.9) leads to diagonalization of the operator

$$\int d\zeta_1^{(-4)} d\zeta_2^{(-4)} \bar{\eta}^{+a}(1)\{\delta^b_a \mathcal{D}_1^{++}\delta^{(1,3)}(1|2) + q^+_a(1)G^{(2,2)}(1|2)q^{+b}(2)\}\eta^+_b(2).$$

Then, the one-loop effective action $\Gamma[V^{++}, q^+]$ defined by the path integral

$$e^{i\Gamma[V^{++},q^+]} = \int \mathcal{D}\bar{\eta}^+\mathcal{D}\eta^+\mathcal{D}\bar{\chi}^{++}\mathcal{D}\chi^{++}e^{iS_2[\eta^+,\bar{\eta}^+,\bar{\chi}^{++},\chi^{++},V^{++},q^+]}$$

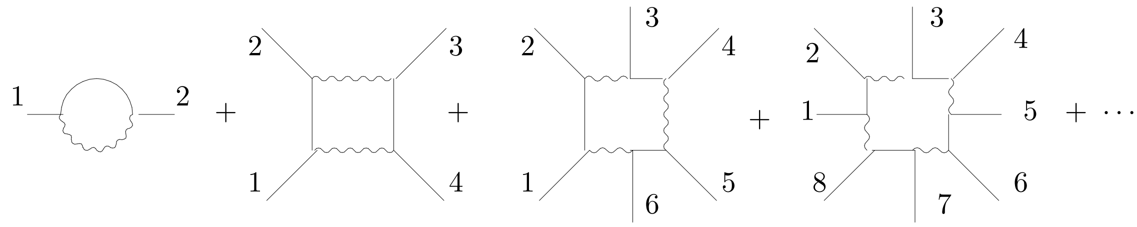
can be formally written as

$$\Gamma[V^{++}, q^+, \breve{q}^+] = i\mathrm{Tr}\ln\{\delta_a^b \mathcal{D}_1^{++}\delta^{(1,3)}(1|2) + q_a^+(1)G^{(2,2)}(1|2)q^{+b}(2)\} + \Gamma[V^{++}], \qquad (3.6)$$

where the last term $\Gamma[V^{++}]$ is a part of the full one-loop effective action which depends only on $\mathcal{N}=2$ gauge superfield. We will study mostly the first term in the above equation since it includes all the hypermultiplet dependence. The expression (3.6) written as an analytic nonlocal superfunctional will be a starting point for our calculations of the one-loop effective action in the hypermultiplet sector. Eq. (3.6) shows that the effective action is well defined in perturbation theory in powers of the non-local interaction $q_a^+(1)G^{(2,2)}(1|2)q^{+b}(2)$. It leads to effective action in the form $\Gamma[V^{++}, \breve{q}^+, q^+] = \sum_{n=1}^{\infty}\Gamma_{2n}[V^{++}, \breve{q}^+, q^+]$. Here the $2n$-th term is given by a supergraph with $2n$ external $\breve{q}^+, q^+$-legs and any number V^{++}-legs. Since $\Gamma[V^{++}, \breve{q}^+, q^+]$ is gauge invariant by construction, one can expect that each coefficient Γ_{2n} depends on background superfield V^{++} only via the strengths $\mathcal{W}, \bar{\mathcal{W}}$ and their covariant derivatives. We emphasize that the supergraphs associated with this procedure contain the background dependent superpropagators.

4. Analysis of supergraphs for hypermultiplet dependent contributions to effective action

The hypermultiplet dependent contributions to the one-loop effective action are presented by following infinite sequence of the supergraphs:



Here the wavy line stands for $\mathcal{N}=2$ gauge superfield propagator and solid external and internal lines stand for background hypermultiplet superfields and quantum hypermultiplet propagators respectively. The numbers $1, 2, ...$ mark the arguments ζ_A, u of the external hypermultiplets lines. As we emphasized above the whole dependence of the contributions on the background gauge superfield is included into background-dependent propagators.

An arbitrary supergraph with $2n$ external hypermultiplet lines looks like a ring consisting of n links of the form $\langle\bar{\eta}^+\eta^+\rangle\langle\chi^{++}\bar{\chi}^{++}\rangle$ or n links of the form $\langle\eta^+\bar{\eta}^+\rangle\langle\bar{\chi}^{++}\chi^{++}\rangle$. The total contribution of these two kinds of the $2n$-point supergraph is given by following general expression

$$i\Gamma_{2n} = \frac{4}{n}\int d\zeta_1^{(-4)}d\zeta_2^{(-4)}\dots d\zeta_{2n}^{(-4)} \times$$
$$\times\frac{(\mathcal{D}_1^+)^4(\mathcal{D}_2^+)^4}{(u_1^+u_2^+)^3}\left\{\frac{1}{\widehat{\Box}_1}\delta_{1|2}^{12}\right\}\frac{(\mathcal{D}_2^+)^4(\mathcal{D}_3^+)^4}{\widehat{\Box}_2\widehat{\Box}_3}(\mathcal{D}_3^{--})^2\{\delta_{2|3}^{12}\delta^{(-2,2)}(u_2,u_3)\}\times$$

$$
\times \frac{(\mathcal{D}_3^+)^4(\mathcal{D}_4^+)^4}{(u_3^+u_4^+)^3}\left\{\frac{1}{\widehat{\Box}_3}\delta^{12}_{3|4}\right\}\ldots\frac{(\mathcal{D}_{2n-2}^+)^4(\mathcal{D}_{2n-1}^+)^4}{\widehat{\Box}_{2n-2}\widehat{\Box}_{2n-1}}(\mathcal{D}_{2n-1}^{--})^2\times
$$
$$
\times\{\delta^{12}_{2n-2|2n-1}\delta^{(-2,2)}(u_{2n-2},u_{2n-1})\}\times
$$
$$
\times\frac{(\mathcal{D}_{2n-1}^+)^4(\mathcal{D}_{2n}^+)}{(u_{2n-1}^+u_{2n}^+)^3}\left\{\frac{1}{\widehat{\Box}_{2n-1}}\delta^{12}_{2n-1|2n}\right\}\frac{(\mathcal{D}_{2n}^+)^4(\mathcal{D}_1^+)^4}{\widehat{\Box}_{2n}\widehat{\Box}_1}(\mathcal{D}_1^{--})^2\{\delta^{12}_{2n|1}\delta^{(-2,2)}(u_{2n},u_1)\}\times
$$
$$
\times\, q_a^+(z_1,u_1)q^{+a}(z_2,u_2)q_b^+(z_3,u_3)\ldots q_c^+(z_{2n-1},u_{2n-1})q^{+c}(z_{2n},u_{2n})\,. \tag{4.1}
$$

Here and further, to avoid coincident harmonic singularities, we keep the $\mathcal{N}=2$ gauge superfield propagator in the form which is manifestly analytic in both argument [29, 8].

The factor $4/n$ has the following origin (see [25]). The contribution from the ring type supergraph composed from n repeating links $\langle\bar\eta^+\eta^+\rangle\langle\chi^{++}\bar\chi^{++}\rangle$ appears with the symmetry factor $2/n$. The same factor $2/n$ arises from the supergraph composed from n repeating links $\langle\eta^+\bar\eta^+\rangle\langle\bar\chi^{++}\chi^{++}\rangle$. Further, each vertex brings the factor $-i$, every $\langle\eta^+\bar\eta^+\rangle$ and $\langle\bar\chi^{++}\chi^{++}\rangle$ propagators contribute the factor i and $i/2$ respectively. Hence total of n links contributes 2^{-n}. Any vertex also carries the coefficient $\sqrt{2}$. This leads to the total factor 2^n. Substituting all these contributions together, we obtain just the coefficient $4/n$.

We begin with direct calculation of the term $\Gamma_2[V^{++},q^{+a}]$ which in the analytic basis reads

$$
i\Gamma_2=\int d\zeta_1^{(-4)}d\zeta_2^{(-4)}du_1du_2\left\{\frac{(\mathcal{D}_1^+)^4(\mathcal{D}_2^+)^4}{(u_1^+u_2^+)^3}\frac{1}{\widehat{\Box}_1}\delta^{12}(1|2)\right\}\times
$$
$$
\times\left\{\frac{(\mathcal{D}_2^+)^4(\mathcal{D}_1^+)^4}{\widehat{\Box}_2\widehat{\Box}_1}\delta^{12}(2|1)(\mathcal{D}_1^{--})^2\delta^{-2,2}(u_2,u_1)\right\}q_a^+(z_1,u_1)q^{+a}(z_2,u_2)\,. \tag{4.2}
$$

According to the general strategy of handling such supergraphs we should first to restore the full Grassmann integration measure at the vertices by rule $d^{12}z_1\,d^{12}z_2=d^{(-4)}\zeta_1\,d^{(-4)}\times\zeta_2(D_1^+)^4\,(D_2^+)^4$. Since we are interested in contributions to the effective action which do not depend on space-time derivatives $\mathcal{D}_mq_a^+$ and spinor derivatives $\mathcal{D}^-_{(\alpha,\dot\alpha)}q_a^+$ of the background hypermultiplets, it is sufficient to impose the following superfields restrictions

$$
\mathcal{D}^-_{(\alpha,\dot\alpha)}q_a^+=0\,,\qquad \mathcal{D}^+_{(\alpha,\dot\alpha)}q^{-a}=0\,. \tag{4.3}
$$

Integrating by parts with the help of delta function we shrink a loop into a point in superspace. However, in harmonic superspace there still remains a non-local expression

$$
i\Gamma_2=\int\frac{dzdu_1du_2}{(u_1^+u_2^+)^3}\frac{(\mathcal{D}_2^+)^4(\mathcal{D}_1^+)^4}{\widehat{\Box}_1^2\widehat{\Box}_2}\delta^{12}(z)|\{(\mathcal{D}_1^{--})^2\delta^{-2,2}(u_2,u_1)\}q_a^+(z_1,u_1)q^{+a}(z_1,u_2)\,. \tag{4.4}
$$

We point out that the presence of the harmonic distribution $(u_1^+u_2^+)^{-3}$ with coincident singularities. If we had the flat covariant derivatives we could use the important identity: $(D_1^+)^4(D_2^+)^4\delta^8(\theta_1-\theta_2)=(u_1^+u_2^+)^4$ and would get a simplification. However, the expression under consideration (4.4) contains the covariant spinor derivatives and further analysis becomes more complicated. In principle, one can use the idea of ref. [34] to express the covariant derivatives $\mathcal{D}_2^{+(\alpha,\dot\alpha)}$ through the covariant derivatives $\mathcal{D}_1^{+(\alpha,\dot\alpha)}$ to evaluate the two-point function of the form $(\mathcal{D}_1^+)^4(\mathcal{D}_2^+)^4\delta^{12}(z_1-z_2)\frac{1}{(u_1^+u_2^+)^q}$. It can help to calculate the Γ_2 (4.4), however a procedure of such a calculation still looks very complicated technically. From our point of view, it is more convenient to start with the representation (4.2) and work in the analytic subspace [16]. Then instead of computing the contribution (4.4) in full $\mathcal{N}=2$ superspace we should actually look for an equivalent expression of the form

$$\int d\zeta^{-4}(D^+)^4\mathcal{L}(\mathcal{W},\bar{\mathcal{W}},q^+)$$

with some $\mathcal{L}$. To do that we return in (4.4) to analytic subspace, use twice the relation [34] and apply the harmonic identities $(u_1^+u_2^+)|_{1=2}=0$, $D_1^{--}(u_1^+u_2^+)=(u_1^-u_2^+)$, $(u_1^+u_2^-)|_{1=2}=1$

$$\frac{1}{(u_1^+u_2^+)^3}(\mathcal{D}_1^+)^4(\mathcal{D}_2^+)^4(\mathcal{D}_1^+)^4=(\mathcal{D}_1^+)^4\{\cdots+(u_1^+u_2^+)(u_2^-u_1^+)^2(u_1^-u_2^+)^2\widehat{\Box}_1\widehat{\Box}_2+\cdots\}\,. \quad (4.5)$$

As a result ones get

$$i\Gamma_2=\int d\zeta^{(-4)}du_1du_2(u_1^+u_2^+)(q_a^+(u_1)q^{+a}(u_2))(D_1^{--})^2\delta^{(-2,2)}(u_2u_1)\frac{(\mathcal{D}^+)^4}{\widehat{\Box}_2}\delta^{12}(z)|. \quad (4.6)$$

In order to be able to take the factor $(D^{--})^2$ off the harmonic delta function and apply the harmonic identities

$$(D_1^{--})^2\delta^{(-2,2)}(u_2u_1)=(D_2^{--})^2\delta^{(2,-2)}(u_2u_1)\,,\qquad D^{--}q^+=q^-\,,\qquad (u_1^+u_2^+)|_{1=2}=0\,,$$

we temporarily restore the full measure of integration. After that we return to the analytic measure. In doing this we keep in mind the restrictions (4.3). Integration over one set of harmonics leads to final result for Γ_2

$$i\Gamma_2=\int d\zeta^{(-4)}du(-4q_a^-q^{+a})\frac{(\mathcal{D}^+)^4}{\widehat{\Box}}\delta^{12}(z)|\,. \quad (4.7)$$

The next step is to calculate the four-leg contribution $\Gamma_4[q^+]$. We start with general relation (4.1) for $n=2$ and perform the same manipulations as in preliminary case. It gives

$$\begin{aligned}
i\Gamma_4 = &\int d\zeta_1^{(-4)}\cdots d\zeta_4^{(-4)}du_1\cdots du_4\frac{(\mathcal{D}_1^+)^4(\mathcal{D}_2^+)^4}{(u_1^+u_2^+)^3}\left\{\frac{1}{\widehat{\Box}_1}\delta^{12}(1|2)\right\}\times\\
&\times\left[\frac{(\mathcal{D}_2^+)^4(\mathcal{D}_3^+)^4}{\widehat{\Box}_2\widehat{\Box}_3}\delta^{12}(2|3)(\mathcal{D}_3^{--})^2\delta^{(-2,2)}(u_2,u_3)\right]\frac{(\mathcal{D}_3^+)^4(\mathcal{D}_4^+)^4}{(u_3^+u_4^+)^3}\left\{\frac{1}{\widehat{\Box}_3}\delta^{12}(3|4)\right\}\times\\
&\times\left[\frac{(\mathcal{D}_4^+)^4(\mathcal{D}_1^+)^4}{\widehat{\Box}_4\widehat{\Box}_1}\delta^{12}(4|1)(\mathcal{D}_1^{--})^2\delta^{(-2,2)}(u_4,u_1)\right]\times\\
&\times q_a^+(z_1,u_1)q^{+a}(z_2,u_2)q_b^+(z_3,u_3)q^{+b}(z_4,u_4)\\
=&\int\frac{dzdu_1\ldots du_4}{(u_1^+u_2^+)^3(u_3^+u_4^+)^3}\frac{(\mathcal{D}_1^+)^4(\mathcal{D}_2^+)^4(\mathcal{D}_3^+)^4(\mathcal{D}_4^+)^4}{\widehat{\Box}_1^2\widehat{\Box}_2\widehat{\Box}_3^2\widehat{\Box}_4}\delta^{12}(z)|(\mathcal{D}_3^{--})^2\delta^{(-2,2)}(u_2,u_3)\times\\
&\times(\mathcal{D}_1^{--})^2\delta^{(-2,2)}(u_4,u_1)q_a^+(1)q^{+a}(2)q_b^+(3)q^{+b}(4)\,. \qquad (4.8)
\end{aligned}$$

Using identities $(\mathcal{D}_3^{--})^2\delta^{(-2,2)}(u_2,u_3) = (\mathcal{D}_2^{--})^2\delta^{(2,-2)}(u_2,u_3)$, and $(\mathcal{D}_2^+)^4(\mathcal{D}_2^{--})^2(\mathcal{D}_3^+)^4 \times \delta^{(2,-2)}(u_2,u_3) = -2\widehat{\Box}_2(\mathcal{D}_2^+)^4$ one obtains

$$
\begin{aligned}
i\Gamma_4 &= \int dz \frac{du_1 du_2}{(u_1^+u_2^+)^6}\frac{(\mathcal{D}_1^+)^4(\mathcal{D}_2^+)^4}{\widehat{\Box}_1^2\widehat{\Box}_2^2}\delta^{12}(z)|q_a^+(1)q^{+a}(2)q_b^+(2)q^{+b}(1) \\
&= \int d\zeta^{(-4)}\frac{du_1du_2}{(u_1^+u_2^+)^2}\frac{(\mathcal{D}_2^+)^4}{\widehat{\Box}_2^2}\delta^{12}(z)D_1^{++}q_a^-(1)q^{+a}(2)q_b^+(2)q^{+b}(1) \\
&= \int d\zeta^{(-4)}du_1du_2[D_1^{--}\delta^{(2,-2)}(u_1u_2)]\frac{(\mathcal{D}_2^+)^4}{\widehat{\Box}_2^2}\delta^{12}(z)q_a^-(1)q^{+a}(2)q_b^+(2)q^{+b}(1).
\end{aligned}
$$

Writing $D_1^{--}\delta^{(-2,2)}(u_1u_2) = D_2^{--}\delta^{(0,0)}(u_1u_2)$ and integrating over u_2 one finally gets

$$
i\Gamma_4 = \frac{1}{2}\int d\zeta^{(-4)}du(-4q^-q^+)^2\frac{(\mathcal{D}^+)^4}{\widehat{\Box}^2}\delta^{12}(z)|\,.
$$

Here we have used the restrictions (4.3).

Analysis of a general term Γ_{2n} is carried out analogously. First of all we transform all analytic subspace integrals into ones over the full superspace taking the factors $(D_k^+)^4(D_{k+1}^+)^4$ from the hypermultiplet propagators. Then we integrate over the sets of Grassmann and space-time coordinates using the corresponding delta-functions in the integrand. It leads to

$$
\begin{aligned}
&\int d^{12}zdu_1\cdots du_{2n}\frac{\delta^{(-2,2)}(u_2,u_3)\delta^{(-2,2)}(u_4,u_5)\cdots\delta^{(-2,2)}(u_{2n},u_1)}{(u_1^+u_2^+)^3(u_3^+u_4^+)^3\ldots(u_{2n-1}^+u_{2n}^+)^3}\times \\
&\times\frac{(\mathcal{D}_2^+)^4(\mathcal{D}_4^+)^4\cdots(\mathcal{D}_{2n}^+)^4}{\widehat{\Box}_1\widehat{\Box}_2\ldots\widehat{\Box}_{2n}}\{\delta^{12}(z-z')|q_a^+(u_1)q^{+a}(u_2)q_b^+(u_3)\ldots q_c^+(u_{2n-1})q^{+c}(u_{2n})\}\,.
\end{aligned}
\tag{4.9}
$$

Then we integrate over $u_2, u_4, \ldots, u_{2n}$ using the harmonic delta functions and obtain

$$
\begin{aligned}
&\int \frac{du_1du_3\cdots du_{2n-1}}{(u_1^+u_3^+)^3(u_3^+u_5^+)^3\ldots(u_{2n-1}^+u_1^+)^3}\frac{(\mathcal{D}_1^+)^4(\mathcal{D}_3^+)^4\ldots(\mathcal{D}_{2n-1}^+)^4}{\widehat{\Box}_1^2\widehat{\Box}_3^2\ldots\widehat{\Box}_{2n-1}^2}\times \\
&\times\Big\{\delta^{12}(z-z')|q_a^+(u_1)q^{+a}(u_3)q_b^+(u_3)q^{+b}(u_5)\ldots q_c^+(u_{2n-1})q^{+c}(u_1)\Big\}\,.
\end{aligned}
\tag{4.10}
$$

After relabelling the indices $c\to a, a\to b,\ldots;\quad 3\to 2,\ldots,(2n-1)\to n$ we get the expression

$$
\begin{aligned}
i\Gamma_{2n} &= \frac{4(-1)^n2^n}{n}\int d^{12}zdu_1\cdots du_n\frac{(\mathcal{D}_1^+)^4(\mathcal{D}_2^+)^4\cdots(\mathcal{D}_n^+)^4}{(u_1^+u_2^+)^3(u_2^+u_3^+)^3\cdots(u_n^+u_1^+)^3}\frac{1}{\widehat{\Box}_1^2\widehat{\Box}_2^2\ldots\widehat{\Box}_n^2}\times \\
&\times\Big\{\delta^{12}(z-z')|_{z=z'}q^{+a}(u_1)q_b^+(u_1)q^{+b}(u_2)q_c^+(u_2)\ldots q_a^+(u_n)\Big\}\,.
\end{aligned}
\tag{4.11}
$$

To simplify the expression (4.11) we represent $q_b^+(u_1)$ as $q_b^+(u_1) = D_1^{++}q_b^-(u_1)$ (since q^+ sits on its mass shell). We note that when acting on $\widehat{\Box}$, D^{++} gives rise to the structures like $(\mathcal{D}^+)^5 = 0$. Now we integrate by parts and remove the harmonic derivative on the harmonic distributions

$$-D_1^{++}\frac{1}{(u_1^+u_2^+)^3(u_n^+u_1^+)^3}=\frac{1}{2}\{(D_1^{--})^2\delta^{(3,-3)}(u_1,u_2)\frac{1}{(u_1^+u_n^+)^3}+(2\leftrightarrow n)\} \tag{4.12}$$

and use the identity [12, 8]

$$(D_1^{--})^2\delta^{(3,-3)}(1|2)=(D_2^{--})^2\delta^{(-1,1)}(1|2)\,. \tag{4.13}$$

Then we take the factor$(D_2^{--})^2$ off the harmonic delta function. It is easy to see that this factor can give a non-vanishing result only when acts on $(\mathcal{D}^+(u_2))^4$. We get

$$\begin{aligned}&\int du_1\ldots(-\frac{1}{2})\delta^{(-1,1)}(1|2)\frac{(\mathcal{D}_1^+)^4(D_2^{--})^2(\mathcal{D}_2^+)^4\ldots(\mathcal{D}_n^+)^4}{(u_2^+u_3^+)^3\ldots(u_n^+u_1^+)^3}\frac{1}{\widehat{\Box}_1^2\widehat{\Box}_2^2\ldots\widehat{\Box}_n^2}\times\\&\times\{\delta^{12}(z)|q^{+a}(u_1)q_b^-(u_1)q^{+b}(u_2)\ldots q_a^+(u_n)\}+(2\leftrightarrow n)\,.\end{aligned} \tag{4.14}$$

We replace $u_2\leftrightarrow u_n$ in the second term here, after that it becomes identical to the first one. Doing integral over u_1 we obtain the expression $(-1)(-2\widehat{\Box}_2)\frac{1}{\widehat{\Box}_2^4}$. At the second step, we repeat the above procedure for $q_c^+(u_2)$, i.e. represent it in the form $q_c^+(u_2)=D_2^{++}q_c^-$ and integrate by parts with respect to $\mathcal{D}_2^{++}$. Performing the same manipulations as above we obtain the factor $(-1)^2(-2\widehat{\Box}_3)^2\frac{1}{\widehat{\Box}_3^6}$.

After $n-4$ analogous steps $q_d^+(u_3)=D_3^{++}q_d^-(u_3)$ we reduce the harmonic integral in expression (4.11) to that over three sets of harmonic:

$$\begin{aligned}&\frac{(\mathcal{D}_u^+)^4(\mathcal{D}_{u_{n-1}}^+)^4(\mathcal{D}_{u_n}^+)^4}{\widehat{\Box}_u^{2n-4}\widehat{\Box}_{u_{n-1}}^2\widehat{\Box}_{u_n}^2}\frac{(2\widehat{\Box})^{n-3}}{(u^+u_{n-1}^+)^3(u_{n-1}^+u_n^+)^3(u_n^+u^+)^3}\times\\&\times\{\delta^{12}(z-z')|\times q^{+a}(u)\ldots q_c^+(u)q^{+c}(u_{n-1})q_d^+(u_{n-1})q^{+d}(u_n)q_a^+(u_n)\}\end{aligned} \tag{4.15}$$

On last step we write $q_c^+(u)=D_u^{++}q_c^-$ and remove the D_u^{++} on the harmonic factor. Repeating the same manipulations we perform the u_{n-1}-integration and get the expression

$$\begin{aligned}i\Gamma_{2n}=&-\frac{(-2)^n2^n}{n}\int d^{12}zdudu_1\frac{(\mathcal{D}_u^+)^4(\mathcal{D}_{u_1}^+)^4}{\widehat{\Box}_u^n\widehat{\Box}_{u_1}^2}\frac{1}{(u^+u_1^+)^6}\times\\&\times\{\delta^{12}(z-z')|q^{+a}(u)q_b^-(u)q^{+b}(u)\ldots q_c^+(u)q^{+c}(u_1)q_a^+(u_1)\}\,.\end{aligned} \tag{4.16}$$

Now ones return to analytic subspace applying the identities (4.5)

$$\int d\zeta^{(-4)}\frac{dudu_1}{(u^+u_1^+)^2}\frac{(\mathcal{D}_u^+)^4}{\widehat{\Box}_u^n}\delta^{12}(z-z')|q^{+a}(u)q_b^-(u)q^{+b}(u)\ldots q_c^+(u)q^{+c}(u_1)q_a^+(u_1)\,.$$

Then we use the identities $q^+(u_1)=(u_1^+u^-)q^+(u)-(u_1^+u^+)q^-$, $q_a^-q^{-a}=0$ leading to the factor $(u_1^+u_2^+)^2$. We get finally

$$i\Gamma_{2n}=\frac{1}{n}\int d\zeta^{(-4)}du\frac{(\mathcal{D}^+)^4}{\widehat{\Box}^n}\delta^{12}(z-z')|(-4q^-q^+)^n\,. \tag{4.17}$$

Here we again have used the restrictions (4.3).

Now sum up all contributions Γ_{2n} (4.17). The result is given in terms of functional determinant of special differential operator:

$$
\begin{aligned}
i\Gamma &= -\int d\zeta^{(-4)} du \ln\left(1 + \frac{4q^-_a q^{+a}}{\widehat{\Box}}\right)(\mathcal{D}^+)^4\delta^{12}(z-z')|_{z'=z} \\
&= -\int d\zeta^{(-4)} du \ln(\widehat{\Box} + 4q^-_a q^{+a})(\mathcal{D}^+)^4\delta^{12}(z-z')|_{z'=z} + \\
&\quad + \int d\zeta^{(-4)} du \ln(\widehat{\Box})(\mathcal{D}^+)^4\delta^{12}(z-z')|_{z'=z}\,.
\end{aligned} \tag{4.18}
$$

It is evident that the second term in (4.18) is (up to the sign) the representation of the one-loop effective action for $\mathcal{N}=4$ SYM theory in sector the $\mathcal{N}=2$ vector multiplet [34, 25]. We see the Γ in (4.18) vanishes when the hypermultiplets vanish. We point out that Γ (4.18) is only a part of full effective action (3.6) which essentially depends on hypermultiplets. The full effective action contains also hypermultiplet independent part $\Gamma(V^{++})$.

We would like to emphasize that in spite of the formal presence of q^-_a in (4.18), the integrand in (4.18) is an analytic superfield in framework of the low-energy approximation (see the restrictions (4.3 in the hypermultiplet sector). Indeed, the combination $q^-_a q^{+a}$ is harmonic-independent [24] and is proportional to $q^{ia}q_{ia}$ where q^{ia} is (constrained) superfield on general $\mathcal{N}=2$ superspace. This quantity should be treated as a constant if we consider only leading low-energy approximation for hypermultiplet dependence (see the restrictions for background hypermultiplet (4.3), which we use in the paper). Such a situation is completely consistent with a general picture of finding the effective action as a series in background field derivatives. If we look for the terms in effective action without derivatives it is sufficient to consider all possible derivatives of background fields to be equal zero. Therefore the eq. (4.18) is completely correct under approximation what was formulated in the paper from the very beginning.[5]

Thus, using the $\mathcal{N}=2$ harmonic superspace formulation of $\mathcal{N}=4$ SYM theory and techniques of harmonic supergraphs we obtained the representation of the one-loop effective action in hypermultiplet sector (4.18). This representation is free of harmonic singularities and, as we will see, admits a straightforward evaluation with help of $\mathcal{N}=2$ superfield heat kernel method. We will show in the next section that the general expression (4.18) allows us to obtain the exact proper-time representation of the effective action and its expansion in covariant spinor derivatives of the $\mathcal{N}=2$ superfield abelian strengths $\mathcal{W}, \bar{\mathcal{W}}$ corresponding to the constant space-time superstrength background

$$
\begin{aligned}
&\mathcal{W}|_{\theta=0} = \text{const}\,, \quad D^\pm_\alpha \mathcal{W}|_{\theta=0} = \text{const}\,, \quad \bar{D}^\pm_{\dot\alpha}\bar{\mathcal{W}}|_{\theta=0} = \text{const}\,, \\
&D^-_\alpha D^+_\beta \mathcal{W}|_{\theta=0} = \text{const}\,, \quad \bar{D}^-_{\dot\alpha}\bar{D}^+_{\dot\beta}\bar{\mathcal{W}}|_{\theta=0} = \text{const}
\end{aligned} \tag{4.19}
$$

and the on-shell background hypermultiplet $q^{\pm a} = q^{ia}u^\pm_i$ where q^{ia} does not depend of harmonics [24] and can be taken to be a constant.

[5] We point out once more that we are considering only the leading derivative independent hypermultiplet contribution to the low-energy effective action. Of course, such an approximation violates the invariance under the hidden $\mathcal{N}=2$ supersymmetry. It would be extremely interesting and useful to develop a systematic procedure to derive the derivative expansion of the effective action in hypermultiplet sector, However this problem is beyond the scope of this paper.

The eq. (4.18) immediately leads to one of the main results concerning the hypermultiplet dependence of one-loop effective action: the hypermultiplet enters to the effective action in the combination $\bar{\mathcal{W}}\mathcal{W} + 2q^-_a q^{+a}$ which is invariant under the R-symmetry of $\mathcal{N} = 4$ supersymmetry for constant $\mathcal{W}$ and on-shell hypermultiplet [24]. To see that we consider the expression $\widehat{\Box} + 4q^-_a q^{+a}$ and use the on-shell form of $\widehat{\Box}$ (3.5). Then $\widehat{\Box} + 4q^-_a q^{+a} = \frac{1}{2}\mathcal{D}^{\alpha\dot{\alpha}}\mathcal{D}_{\alpha\dot{\alpha}} - \frac{i}{\sqrt{2}}((D^{+\alpha}\mathcal{W})\mathcal{D}^-_\alpha + (\bar{D}^+_{\dot{\alpha}}\bar{\mathcal{W}})\bar{\mathcal{D}}^{-\dot{\alpha}}) + 2\mathcal{W}\bar{\mathcal{W}} + 4q^-_a q^{+a}$. We see that this operator contain just the above combination (two last terms without derivatives) and just this combination will be on of the arguments of hypermultiplet dependent effective action (36) in the low-energy approximation.[6]

5. The proper-time representation of the effective action

Relation (4.18) possesses remarkable features. First, we started with the model of two interacting fields V^{++}, q^+ and resummed the supergraphs by such a way that the effective action is expressed in terms of a differential operator acting only in the sector of the vector multiplet. The whole dependence on the hypermultiplets is included in this operator. Second, the effective action is written as an integral over the analytic subspace of the harmonic superspace. It is co-ordinated with the classical action of the theory which is also written as an integral over the analytic subspace. Third, the relation (4.18) has the form $\mathrm{Tr}\ln\hat{A}$ with the operator $\hat{A} = \widehat{\Box} + 4q^-_a q^{+a}$ acting on the analytic superfields. We emphasize that the above simple form of the one-loop effective action is not evident from the very beginning, it is the result of resummation of the infinite sequence of the one-loop harmonic supergraphs with arbitrary number hypermultiplet external legs. The form of the effective action (4.18) is basic for using the proper-time representation:

$$\begin{aligned}\Gamma &= i\int d\zeta^{(-4)}du\int_0^\infty \frac{ds}{s}e^{-s(\widehat{\Box}+4q^-_a q^{+a})}(\mathcal{D}^+)^4\delta^{12}(z-z')|_{z'=z}\\ &= i\int_0^\infty \frac{ds}{s}\mathrm{Tr}\left\{K(s)e^{-s(4q^-_a q^{+a})}\right\}.\end{aligned} \tag{5.1}$$

Here $K(s)$ is a superfield heat kernel, the operation Tr means the functional trace in the analytic subspace of the harmonic superspace $\mathrm{Tr}K(s) = \mathrm{tr}\int d\zeta^{(-4)}K(\zeta,\zeta|s)$, where tr denotes the trace over the discrete indices. As a result, the problem of finding the one-loop effective action is reduced to evaluation of the kernel $K(s) = e^{-s\widehat{\Box}}$. Further it is convenient to deal with $\widehat{\Box}$ written by a definition as[7]

$$\widehat{\Box} = \frac{1}{2}\mathcal{D}^{\dot{\alpha}\alpha}\mathcal{D}_{\alpha\dot{\alpha}} + \frac{i}{2}(D^{+\alpha}\mathcal{W})\mathcal{D}^-_\alpha + \frac{i}{2}(\bar{D}^+_{\dot{\alpha}}\bar{\mathcal{W}})\bar{\mathcal{D}}^{-\dot{\alpha}} + \mathcal{W}\bar{\mathcal{W}}\,. \tag{5.2}$$

[6]It is worth pointing out that such an approximation violates R-symmetry of initial classical model. Problem of constructing the general derivative expansion of effective action containing the hypermultiplet spinor derivatives and preserving the R-symmetry is open at present. We consider this work as a first step towards this generic problem.

[7]Note that the replacements $\mathcal{W} \to \frac{\mathcal{W}}{-\sqrt{2}}$, $\bar{\mathcal{W}} \to \frac{\bar{\mathcal{W}}}{-\sqrt{2}}$ reduces the expression (5.2) to (3.5).

For the effective action calculation (5.1) we will use the techniques of symbols of operators in the analytic subspace of $\mathcal{N}=2$ harmonic superspace.[8] We begin with the Fourier representation of the delta-function in $\mathcal{N}=2$ superspace

$$\delta^{12}(z-z') = \int \frac{d^4p}{(2\pi)^4} \int d^4\psi^+ d^4\psi^- e^{ip_m(x-x')^m} \times \\ \times e^{(\theta-\theta')^{+\alpha}\psi^-_\alpha + \bar\psi^-_{\dot\alpha}(\bar\theta-\bar\theta')^{+\dot\alpha}} e^{(\theta-\theta')^{-\alpha}\psi^+_\alpha + \bar\psi^+_{\dot\alpha}(\bar\theta-\bar\theta')^{-\dot\alpha}}, \tag{5.3}$$

where $p_M = \{p_m, \psi^\pm_\alpha, \bar\psi^\pm_{\dot\alpha}\}$ is a cotangent supervector at a superspace point z and $d^4\psi^+ = \frac{1}{16}d^2\psi^+ d^2\bar\psi^+$. As a result, the heat kernel at coincident points takes the form

$$K(\zeta,\zeta|s) = \int \frac{d^4p}{(2\pi)^4} d^8\psi \exp\Big\{ -s\Big(\frac{1}{2}(\mathcal{D}+ip)^{\dot\alpha\alpha}(\mathcal{D}+ip)_{\alpha\dot\alpha} + \frac{i}{2}(D^{+\alpha}\mathcal{W})(\mathcal{D}+\psi)^-_\alpha + \\ + \frac{i}{2}(\bar D^+_{\dot\alpha}\bar{\mathcal{W}})(\bar{\mathcal{D}}+\bar\psi)^{-\dot\alpha} + \mathcal{W}\bar{\mathcal{W}}\Big)\Big\}(\mathcal{D}^+ + \psi^+)^4 \times \mathbf{1}. \tag{5.4}$$

Then, we have to act by the operators in the exponential to the right on the unity. After that, all differential operators can act only on $\mathcal{W}, \bar{\mathcal{W}}$. Therefore, the final result is expressed in terms of strengths $\mathcal{W}, \bar{\mathcal{W}}$ and their spinor derivatives. This procedure can be organized, for example, as follows. We extract in (5.4) the exponential of the leading symbol $\frac{1}{2}p^{\alpha\dot\alpha}p_{\alpha\dot\alpha}$ of the operator (5.2) and expand the rest exponential in a power series in covariant derivatives. Then we move all the derivatives to the right, commuting or anticommuting them with the coefficients of the operator (5.2), act on unity and kill them. Last step is gaussian integration over momenta p and trivial integration over odd variables ψ. The result will be a series in covariant derivatives of the strengths. However, this way of consideration demands a lot of straightforward calculations to get a final result in manifest covariant form.

A method generating the manifest supersymmetric asymptotical expansion of heat kernel in $\mathcal{N}=1$ superspace has been developed in [26, 28, 27]. We generalize this method for the heat kernel in $\mathcal{N}=2$ harmonic superspace. In each superspace point we introduce a tangent space forming the normal-coordinate system and a fiber frame obtained by a parallel transport from the base point. The pseudodifferential operators can be reexpressed in this local representation of the vector bundle. We consider the heat kernel using these operators and derive an algorithm for the asymptotic expansions of the heat kernel.

Let us introduce the following notions

$$A^{+\alpha} = \frac{i}{2}[D^{+\alpha},\mathcal{W}], \quad \bar A^{+\dot\alpha} = -\frac{i}{2}[\bar D^{+\dot\alpha},\bar{\mathcal{W}}], \quad A^{-\alpha} = [D^{-\alpha},\mathcal{W}], \quad \bar A^{-\dot\alpha} = [\bar D^{-\dot\alpha},\bar{\mathcal{W}}], \\ \{D^-_\alpha, A^+_\beta\} = N_{\alpha\beta} = N_{\beta\alpha} = \frac{i}{2}D^-_\alpha D^+_\beta \mathcal{W}, \quad \{\bar D^-_{\dot\alpha}, \bar A^+_{\dot\beta}\} = \bar N_{\dot\alpha\dot\beta} = \bar N_{\dot\beta\dot\alpha} = -\frac{i}{2}\bar D^-_{\dot\alpha}\bar D^+_{\dot\beta}\bar{\mathcal{W}}. \tag{5.5}$$

In this terms the algebra of covariant derivatives (2.4) takes the form:

$$[\mathcal{D}_{\alpha\dot\alpha}, x^{\beta\dot\beta}] = 2\delta^{\beta\dot\beta}_{\alpha\dot\alpha}, \quad \{\mathcal{D}^-_\alpha, \theta^{+\beta}\} = \delta^\beta_\alpha, \quad \{\bar{\mathcal{D}}^-_{\dot\alpha}, \bar\theta^{+\dot\beta}\} = \delta^{\dot\beta}_{\dot\alpha},$$

[8] Applications of techniques of symbols of operators for calculating the effective action in $\mathcal{N}=1$ superspace are given in [28].

$$
\begin{aligned}
[\mathcal{D}^{\dot\alpha\alpha}, \mathcal{D}_{\beta\dot\beta}] &= -(\delta^{\dot\alpha}_{\dot\beta} N^{\alpha}_{\beta} + \delta^{\alpha}_{\beta} \bar N^{\dot\alpha}_{\dot\beta}) = -F^{\alpha\dot\alpha}_{\beta\dot\beta}\,, \\
[\mathcal{D}^{-}_{\alpha}, \mathcal{D}_{\beta\dot\beta}] &= \varepsilon_{\alpha\beta} \bar A^{-}_{\dot\beta}, \quad [\bar{\mathcal{D}}^{-}_{\dot\alpha}, \mathcal{D}_{\beta\dot\beta}] = \varepsilon_{\dot\alpha\dot\beta} A^{-}_{\beta}\,, \\
\mathcal{D}_m A^{\pm}_{(\alpha,\dot\alpha)} &= \mathcal{D}^{\pm}_{(\delta,\dot\delta)} N_{\alpha\beta} = \mathcal{D}^{\pm}_{(\delta,\dot\delta)} \bar N_{\dot\alpha\dot\beta} = \mathcal{D}_m N_{\alpha\beta} = \mathcal{D}_m \bar N_{\dot\alpha\dot\beta} = 0\,.
\end{aligned}
\tag{5.6}
$$

We see that the set of covariant derivatives together with the on-shell background superfields $\mathcal{W}, \bar{\mathcal{W}}$, corresponding to the constant space-time configurations (4.19), and their low-order derivatives generates a finite dimensional Lie superalgebra (5.5)–(5.6).

At the next step, we lift the shifted operators $X_m = \mathcal{D}_m + ip_m, X^{\pm}_{(\alpha,\dot\alpha)} = \mathcal{D}^{\pm}_{(\alpha,\dot\alpha)} + \psi^{\pm}_{(\alpha,\dot\alpha)}$ satisfying the same algebra (5.5)-(5.6) in the tangent space by the "exponential map" $X(p_M, \partial/\partial p_M) = U^{-1} X_M U$[9] with

$$
U = e^{-\partial^{-\alpha}_{\psi}\mathcal{D}^{+}_{\alpha} - \bar{\mathcal{D}}^{+}_{\dot\alpha}\bar\partial^{-\dot\alpha}_{\psi}} \cdot e^{2\theta^{-\alpha} p_{\alpha\dot\alpha}\bar\partial^{+\dot\alpha}_{\psi} - 2\partial^{+\alpha}_{\psi} p_{\alpha\dot\alpha}\bar\theta^{-\dot\alpha}} \cdot e^{\partial^{+\alpha}_{\psi}\mathcal{D}^{-}_{\alpha} + \bar{\mathcal{D}}^{-}_{\dot\alpha}\bar\partial^{+\dot\alpha}_{\psi}} \cdot e^{-\frac{i}{2}\partial^{\dot\alpha\alpha}_{p}\mathcal{D}_{\alpha\dot\alpha}}\,,
$$

where the role of a tangent vectors forming a normal-coordinate frame plays a set of derivatives

$$
\partial^{\mp\alpha}_{\psi} \equiv \frac{\partial}{\partial\psi^{\pm}_{\alpha}}, \quad \bar\partial^{\mp\dot\alpha}_{\psi} \equiv \frac{\partial}{\partial\bar\psi^{\pm}_{\dot\alpha}}\,, \qquad \partial^{\alpha\dot\alpha}_{p} \equiv \frac{\partial}{\partial p_{\alpha\dot\alpha}} \qquad [\partial^{M}, p_N\} = \delta^{M}_{N}\,.
$$

The action of the operator U on the shifted operators X_M is given by following expressions

$$
\begin{aligned}
U^{-1}(X^{+}_{\alpha})U &= \psi^{+}_{\alpha}, \qquad U^{-1}(\bar X^{+}_{\dot\alpha})U = -\bar\psi^{+}_{\dot\alpha}\,, \\
U^{-1}(X^{-}_{\alpha})U &= -\psi^{-}_{\alpha} + 2\bar\partial^{-\dot\alpha}_{\psi} p_{\alpha\dot\alpha} + \mathcal{O}(\partial^{-}_{\psi}, \partial_p)\,, \quad U^{-1}(\bar X^{-}_{\dot\alpha})U = \bar\psi^{-}_{\dot\alpha} + 2\partial^{-\alpha}_{\psi} p_{\alpha\dot\alpha} + \mathcal{O}(\partial^{-}_{\psi}, \partial_p)\,, \\
\{X^{+}_{\alpha}, \bar X^{-}_{\dot\alpha}\} &= 2X_{\alpha\dot\alpha} = -\{X^{-}_{\alpha}, \bar X^{+}_{\dot\alpha}\}\,, \\
X_{\alpha\dot\alpha} &= ip_{\alpha\dot\alpha} + \partial^{+}_{\psi\alpha}(\bar D^{-}_{\dot\alpha}\bar{\mathcal{W}}) - \bar\partial^{+}_{\psi\dot\alpha}(D^{-}_{\alpha}\mathcal{W}) - \frac{1}{8}\{\partial^{\dot\beta}_{p\alpha} + \\
&\quad + (\bar D^{+}_{\dot\beta}\bar D^{-}_{\dot\alpha}\bar{\mathcal{W}}) + \partial^{\beta}_{p\dot\alpha}(D^{-}_{\beta} D^{+}_{\alpha}\mathcal{W}\} + \mathcal{O}(\partial^{-}_{\psi}, \partial_p)\,.
\end{aligned}
\tag{5.7}
$$

The map of the function on the superspace in the point z into the tangent superspace is given by

$$
\begin{aligned}
\mathcal{W} &\to \mathcal{W} - \partial^{+\alpha}_{\psi}(D^{-}_{\alpha}\mathcal{W}) + \mathcal{O}(\partial^{-}_{\psi}, \partial_p), \quad \bar{\mathcal{W}} \to \bar{\mathcal{W}} + \bar\partial^{+\dot\alpha}_{\psi}(\bar D^{-}_{\dot\alpha}\bar{\mathcal{W}}) + \mathcal{O}(\partial^{-}_{\psi}, \partial_p)\,, \\
D^{+}_{\alpha}\mathcal{W} &\to D^{+}_{\alpha}\mathcal{W} - \partial^{+\beta}_{\psi}(D^{-}_{\beta} D^{+}_{\alpha}\mathcal{W}) + \mathcal{O}(\partial^{-}_{\psi}, \partial_p)\,, \\
\bar D^{+}_{\dot\alpha}\bar{\mathcal{W}} &\to \bar D^{+}_{\dot\alpha}\bar{\mathcal{W}} + \bar\partial^{+\dot\beta}_{\psi}(\bar D^{-}_{\dot\beta}\bar D^{+}_{\dot\alpha}\bar{\mathcal{W}}) + \mathcal{O}(\partial^{-}_{\psi}, \partial_p)\,, \\
D^{-}_{\alpha}\mathcal{W} &\to D^{-}_{\alpha}\mathcal{W} + \mathcal{O}(\partial^{-}_{\psi}, \partial_p), \quad \bar D^{-}_{\dot\alpha}\bar{\mathcal{W}} \to \bar D^{-}_{\dot\alpha}\bar{\mathcal{W}} + \mathcal{O}(\partial^{-}_{\psi}, \partial_p)\,.
\end{aligned}
\tag{5.8}
$$

We point out that in this representation the operators X, superfields and their derivatives satisfy the same algebra (5.6), (5.6). Generally speaking, if the background superfields are arbitrary, all above quantities are presented by an infinite series over ∂_p and a finite series over Grassmann derivatives ∂_ψ with the coefficients in a fixed point z^A. But for the considered background (on-shell and space-time constant background fields) the representation (5.7), (5.8) is exact.

[9]Main property of this transformation is to eliminate the operators $\mathcal{D}_m, \mathcal{D}^{+}_{\alpha}, \mathcal{D}^{-}_{\dot\alpha}$ in X_M. Further we use the same notation X_M for both transformed and initial quantities.

The actual calculation of the effective action (5.1) with the kernel (5.4) is based on the following observation. The operator $e^{-s\widehat{\Box}_U}$, where the operator $\widehat{\Box}_U$ is given in terms of shifted variables X (5.7), can be considered as an evolution operator for a Bose-Fermi quantum system with a hamiltonian $\hat{H} = \widehat{\Box}_U$. Eqs. (5.2), (5.7) show that the hamiltonian $\hat{H}$ is a quadratic form in the operators $p, \partial_p, \psi, \partial_\psi$ with constant coefficients (due to background under consideration). Therefore, calculation of $TrK(s) = \int d\zeta^{-4} K(\zeta, \zeta|s)$ with $K(\zeta, \zeta|s)$ given by (5.4), is an exactly solvable problem.

Let us return to Eq. (5.4) where all operators and fields along with their derivatives are written in the representation (5.7). This is equivalent to an extension of the heat kernel to the tangent bundle on the superspace point z where the coordinate z is considered as a constant parameter. According to (5.4) the evolution operator should act on the unity. It is evident that a result of such action is obtained if we recommute all derivatives with p and ψ to the right and omit them. This procedure can be realized on the base of the Baker-Campbell-Hausdorff formula corresponding to the algebra (5.6), (5.6) (see appendix A). The result we obtain is called a symbol of the evolution operator. This symbol has to be integrated over bosonic p and fermionic ψ variables, which leads us to the heat kernel (5.4). Next useful observation is based on the fact that the exponential in the evolution operator contains only $\mathcal{D}^- \sim \psi^-$. All $\mathcal{D}^+ \sim \psi^+$ are in the pre-exponential factor $(\mathcal{D}^{+4})$ and saturate the integral over $d^4\psi^+$. Therefore we must omit all $\mathcal{O}(\partial^-_\psi)$ in the operator $\hat{H}$ that gives us a more simple expression for (5.7), (5.8). In addition, to do the Berezin integral over ψ^- sector we have to extract a "projector" $(\psi^-)^4$ from the exponential under consideration.

This program can be effectively realized if we present the exponent $K(s) = e^{-s\widehat{\Box}}$ as a products of several operator exponents. Such a construction allows to overcome the difficulties which arose in previous attempts of computing the effective action of $\mathcal{N} = 2$ SYM theories directly in $\mathcal{N} = 2$ superspace. We write the operator $K(s) = e^{-s\widehat{\Box}}$ in the form

$$
\begin{aligned}
K(s) &= \exp\left(-s\left\{A^+\mathcal{D}^- + \bar{A}^+\bar{\mathcal{D}}^- + \frac{1}{2}\mathcal{D}^{\dot{\alpha}\alpha}\mathcal{D}_{\alpha\dot{\alpha}} + \mathcal{W}\bar{\mathcal{W}}\right\}\right) \\
&= \exp\{-f_{\alpha\dot{\alpha}}(s)\mathcal{D}^{\dot{\alpha}\alpha}\}\exp\left\{-s\frac{1}{2}\mathcal{D}^{\dot{\alpha}\alpha}\mathcal{D}_{\dot{\alpha}\alpha}\right\}\exp\{-\Omega(s)\}\exp\{-s(A^{+\alpha}\mathcal{D}^-_\alpha + \bar{A}^{+\dot{\alpha}}\bar{\mathcal{D}}^-_{\dot{\alpha}})\}
\end{aligned}
\tag{5.9}
$$

with some unknown coefficients in the right hand side. These coefficients can be found directly, i.e. using the Baker-Campbell-Hausdorff formula (representation of the Baker-Campbell-Hausdorff formula we use is given in appendix A), and by solution to the system of a differential equation on the coefficients. Both ways lead to the same results. To find the mentioned system of equations we consider $(\frac{d}{ds}K)K^{-1}$ and substitute for K first and second lines in (5.9) subsequently.

Equations for the functions $f^{\dot{\alpha}\alpha}(s)$ have the form

$$
\begin{aligned}
\dot{f}_{\alpha\dot{\alpha}}(s) = &-f_{\beta\dot{\beta}}F^{\dot{\beta}\beta}_{\dot{\alpha}\alpha} - A^{+\beta}(D^-_\beta f_{\alpha\dot{\alpha}}) - \bar{A}^{+\dot{\beta}}(\bar{D}^-_{\dot{\beta}} f_{\alpha\dot{\alpha}}) + \\
&+ A^+_\beta \bar{A}^-_{\dot{\beta}}\left(\int_0^s d\tau e^{\tau F}\right)^{\dot{\beta}\beta}_{\dot{\alpha}\alpha} + \bar{A}^+_{\dot{\beta}} A^-_\beta\left(\int_0^s d\tau e^{\tau F}\right)^{\dot{\beta}\beta}_{\dot{\alpha}\alpha}.
\end{aligned}
\tag{5.10}
$$

Analogously, equation for the function Ω is

$$\dot{\Omega}(s) - \mathcal{W}\bar{\mathcal{W}} = -A^{+\alpha}(D^-_\alpha\Omega) - \bar{A}^{+\dot\alpha}(\bar{D}^-_{\dot\alpha}\Omega) + A^+_\alpha f^{\alpha\dot\alpha}\bar{A}^-_{\dot\alpha} + \bar{A}^+_{\dot\alpha} f^{\dot\alpha\alpha}A^-_\alpha - \\ -\frac{1}{2}A^+_\beta\bar{A}^-_{\dot\beta}\left(\int_0^s d\tau e^{-\tau F}\right)^{\dot\beta\beta}_{\dot\alpha\alpha} F^{\dot\alpha\alpha}_{\dot\rho\rho}f^{\dot\rho\rho} - \frac{1}{2}\bar{A}^+_{\dot\beta}A^-_\beta\left(\int_0^s d\tau e^{-\tau F}\right)^{\dot\beta\beta}_{\dot\alpha\alpha} F^{\dot\alpha\alpha}_{\dot\rho\rho}f^{\dot\rho\rho}. \tag{5.11}$$

It is easy to show that the solution to equation (5.10) can be written as

$$f_{\alpha\dot\alpha} = -A^+_\delta\mathcal{N}^{\delta\dot\delta}_{\alpha\dot\alpha}(s)\bar{A}^-_{\dot\delta} - \bar{A}^+_{\dot\delta}\bar{\mathcal{N}}^{\dot\delta\delta}_{\dot\alpha\alpha}(s)A^-_\delta\,, \tag{5.12}$$

where the functions $\mathcal{N}(N,\bar{N}), \bar{\mathcal{N}}(N,\bar{N})$ are listed in the appendix B. Solution of the equation (5.11) has the form

$$\Omega(s) = s\mathcal{W}\bar{\mathcal{W}} + A^{+\alpha}\Omega^-_\alpha(s) + \bar{A}^{+\dot\alpha}\bar{\Omega}^-_{\dot\alpha}(s) + (A^+)^2\Psi^{(-2)}(s) + (\bar{A}^+)^2\bar{\Psi}^{(-2)}(s) + \\ + A^{+\alpha}\bar{A}^+_{\dot\alpha}\Psi^{\dot\alpha(-2)}_\alpha(s)\,. \tag{5.13}$$

We point out that this solution is a finite order polynomials in powers of the grassmannian elements $A^\pm$, $\bar{A}^\pm$ (5.5). The coefficients $\Omega^-_\alpha(s)$, $\bar{\Omega}^-_{\dot\alpha}(s)$, $\Psi^{(-2)}(s)$, $\bar{\Psi}^{(-2)}(s)$, $\Psi^{\dot\alpha(-2)}_\alpha(s)$ are given in the appendix B.

Now it is useful to write the last exponential in (5.9) in the form

$$e^{-s(A^+\mathcal{D}^- + \bar{A}^+\bar{\mathcal{D}}^-)} = 1 + a^{+\alpha}\mathcal{D}^-_\alpha + \bar{a}^{+\dot\alpha}\mathcal{D}^-_{\dot\alpha} + f^{+2}(\mathcal{D}^-)^2 + \bar{f}^{+2}(\bar{\mathcal{D}}^-)^2 + f^{+2\dot\alpha\alpha}\mathcal{D}^-_\alpha\bar{\mathcal{D}}^-_{\dot\alpha} + \\ + \bar{\Xi}^{+3\dot\alpha}\bar{\mathcal{D}}^-_{\dot\alpha}(\mathcal{D}^-)^2 + \Xi^{+3\alpha}\mathcal{D}^-_\alpha(\bar{\mathcal{D}}^-)^2 + \Omega^{+4}(\mathcal{D}^-)^2(\bar{\mathcal{D}}^-)^2\,. \tag{5.14}$$

The coefficients of this expansion can be found exactly and are given in the appendix B. We specially point out that $\Omega^{+4}(s) \sim (A^+)^4$. The integrand of the kernel (5.4) is represented as a product of the Schwinger type kernel $e^{-s\frac{1}{2}\mathcal{D}^{\alpha\dot\alpha}\mathcal{D}_{\alpha\dot\alpha}}$ and terms of expansion (5.14) in powers of $\mathcal{D}^-$ as well as of expansion $e^{-f_{\alpha\dot\alpha}\mathcal{D}^{\alpha\dot\alpha}}$, $e^{-\Omega(s)}$ in powers of Grassmann variables $A^+, \bar{A}^+$. It leads to

$$K(s) = \int\frac{d^4p}{(2\pi)^4}d^8\psi e^{-s\frac{1}{2}X^{\alpha\dot\alpha}X_{\alpha\dot\alpha}}\left\{1 + \frac{1}{2}f_{\alpha\dot\alpha}(s)X^{\alpha\dot\alpha}f_{\beta\dot\beta}(s)X^{\beta\dot\beta} + \cdots\right\}\times \\ \times e^{-\Omega(s)}\left\{1 + \cdots + \Omega^{+4}(s)(\psi^-)^4\right\}(\psi^+)^4\times\mathbf{1}\,. \tag{5.15}$$

This very complicated expression can be significantly simplified using remarkable properties of the Berezin integral. Only last term in the last braces gives a nontrivial contribution. Since $\Omega^{+4} \sim (A^+)^4$ and all $f_{\alpha\dot\alpha}$, Ω are constructed from the elements $A^+, \bar{A}^+$ we must omit in the above expression for $K(s)$ all this functions except $e^{-s\mathcal{W}\bar{\mathcal{W}}}$. As a result we have

$$K(s) = \int\frac{d^4p}{(2\pi)^4}K_{Sch}(s)e^{-s\mathcal{W}\bar{\mathcal{W}}}\Omega^{+4}(s)\,. \tag{5.16}$$

Last step of the consideration is computing the Schwinger type kernel for the operator $\Box_{\rm cov}(X_m) = \frac{1}{2}X^{\alpha\dot\alpha}X_{\alpha\dot\alpha}$, $K_{Sch}(s) \equiv \int\frac{d^4p}{(2\pi)^4}\cdot e^{-s\Box_{\rm cov}(X_m)}$, where the operators $X_{\alpha\dot\alpha}$ are given in (5.7). Such a computation is standard now (see e.g. [26] for details).[10] We write down only the final result

$$K_{Sch}(s) = \frac{i}{(4\pi s)^2} \frac{s^2(N^2 - \bar{N}^2)}{\cosh(sN) - \cosh(s\bar{N})} \, . \tag{5.17}$$

Here N is given by $N = \sqrt{-\frac{1}{2}D^4\mathcal{W}^2}$. It can be expressed in terms of the two invariants of the abelian vector field $\mathcal{F} = \frac{1}{4}F^{mn}F_{mn}$ and $\mathcal{G} = \frac{1}{4}{}^{\star}F^{mn}F_{mn}$ as $N = \sqrt{2(\mathcal{F} + i\mathcal{G})}$. In the context of $\mathcal{N} = 4$ SYM theory this kernel has been found in [35, 5] on the base of various approaches. Here, we derived the kernel (5.17) completely in terms of $\mathcal{N} = 2$ harmonic superspace.

6. Effective action and its spinor covariant derivative expansion

In the previous section we developed the proper-time techniques in the harmonic superspace. Now we apply this techniques to construct the effective action.

Effective action is written in the form (5.1) where the heat kernel at coincident points is given by (5.4). We apply the decomposition (5.16), (5.17) for $K_{Sch}(s)$ (B.17) for $\Omega^{(+4)}$ and take into account that (i) Eq. (5.4) already contains $(\psi^+)^4$ and hence we can use immediately $\int d^4\psi^+(\psi^+)^4 = 1$, (ii) to saturate the integration over ψ^- it is sufficient to keep in eq. (5.14) only the last term $\Omega^{+4}(\psi^-)^4$, (iii) since $\Omega^{+4} \sim (A^+)^4$ (B.17) we have to omit in (5.9) all terms which dependent on A^+. It leads to the effective action in the final form

$$\Gamma = \frac{1}{(4\pi)^2} \int d\zeta^{(-4)} du \int_0^\infty \frac{ds}{s^3} e^{-s(2\mathcal{W}\bar{\mathcal{W}} + 4q_a^- q^{+a})} \frac{s^2(N^2 - \bar{N}^2)}{\cosh(sN) - \cosh(s\bar{N})} \times$$
$$\times \frac{1}{16}(D^+\mathcal{W})^2(\bar{D}^+\bar{\mathcal{W}})^2 \frac{\cosh(sN) - 1}{N^2} \frac{\cosh(s\bar{N}) - 1}{\bar{N}^2} \, . \tag{6.1}$$

One can show that the integrand in (6.1) can be expended in power series in the quantities $s^2N^2, s^2\bar{N}^2$. After change of proper time s to $s' = s2\mathcal{W}\bar{\mathcal{W}}$ we get the expansion in powers of $s'^2\frac{4N^2}{4(\mathcal{W}\bar{\mathcal{W}})^2}$ and their conjugate. Besides, we point out, since the integrand of (6.1) is already $\sim (A^+)^4$ we can change in each term of expansion the quantities $N^2, \bar{N}^2$ by superconformal invariants Ψ^2 and $\bar{\Psi}^2$ [5] expressing these quantities from

$$\bar{\Psi}^2 = \frac{1}{\bar{\mathcal{W}}^2} D^4 \ln \mathcal{W} = \frac{1}{2\bar{\mathcal{W}}^2} \left\{ \frac{N_\alpha^\beta N_\beta^\alpha}{\mathcal{W}^2} + 4\frac{A^{+\alpha}N_\alpha^\beta A_\beta^-}{\mathcal{W}^3} + 3\frac{(A^+)^2(A^-)^2}{\mathcal{W}^4} \right\} \tag{6.2}$$

and its conjugate. After that, one can show that each term of the expansion can be rewritten as an integral over general $\mathcal{N} = 2$ superspace.

The expression (6.1) at vanishing hypermultiplets has been obtained in refs. [5, 34] by other methods. Hypermultiplet dependent effective action was derived in [27] in terms $\mathcal{N} = 1$ superfields, its transformation to $\mathcal{N} = 2$ harmonic form has been done in [27]

[10] We note that taking the operation tr in the Lorentz indices is trivial due to the identity

$$N_\alpha^\beta N_\beta^\delta = -\frac{1}{4}D_\alpha^- D^{+\beta}\mathcal{W} D_\beta^- D^{+\delta}\mathcal{W} = -\frac{1}{8}D_\alpha^- D^{+\beta} D_\beta^- D^{+\delta}\mathcal{W}^2 = -\frac{1}{2}\delta_\alpha^\delta (D)^4\mathcal{W}^2 = \delta_\alpha^\delta N^2$$

on the base of a heuristic prescription how to reconstruct the effective action, given in terms of $\mathcal{N}=1$ superfields, in manifest $\mathcal{N}=2$ supersymmetric form. Now we justify the prescription used in [27] and derive the hypermultiplet depended effective action (6.1) completely in terms of harmonic superspace.

Now we give a few first terms of the effective action expansion (6.1) in a power series in $N^2, \bar{N}^2$ and compare them with the results of $\mathcal{N}=1$ superspace calculations under the special prescription about reconstruction of manifest $\mathcal{N}=2$ supersymmetric form of the effective written in terms of $\mathcal{N}=1$ superfields. It was done in the work [27] on the base of a spinor covariant derivative expansion of the effective action. We will see that spinor derivative expansion of (6.1) actually reproduces the expansion in [27]. Each term of the spinor covariant derivative expansion of the effective action contains a definite power of the abelian strength F_{mn}. Such an expansion allows to extract an explicitly dependence of q^-q^+. Since $N^2, \bar{N}^2$ includes the spinor covariant derivatives of superstrengths (see (5.5), this is just expression in spinor covariant derivatives of $\mathcal{W}, \bar{\mathcal{W}}$. We use the expansions of $K_{Sch}(s)$ and $\Omega^{+4}(s)$. It leads to

$$\Gamma = \frac{1}{(8\pi)^2}\int d\zeta^{(-4)}du\int_0^\infty ds\cdot se^{-s(2\mathcal{W}\bar{\mathcal{W}}+4q^-q^+)}D^+\mathcal{W}D^+\mathcal{W}\bar{D}^+\bar{\mathcal{W}}\bar{D}^+\bar{\mathcal{W}}\times \\ \times\left\{1+\frac{s^4}{2\cdot 5!}D^4\mathcal{W}^2\bar{D}^4\bar{\mathcal{W}}^2+\cdots\right\}. \tag{6.3}$$

Leading low-energy correction corresponds to F^4-term. Performing the integral over s we have

$$\begin{aligned}\Gamma_{F^4} &= \frac{1}{(4\pi)^2}\int d\zeta^{(-4)}du\frac{1}{16}\frac{D^+\mathcal{W}D^+\mathcal{W}}{\mathcal{W}^2}\frac{\bar{D}^+\bar{\mathcal{W}}\bar{D}^+\bar{\mathcal{W}}}{\bar{\mathcal{W}}^2}\frac{1}{(1-(-\frac{2q^-q^+}{\mathcal{W}\bar{\mathcal{W}}}))^2}\\ &= \sum_{k=0}^{\infty}\frac{1}{(4\pi)^2}\int d\zeta^{(-4)}du\frac{1}{16}\frac{D^+\mathcal{W}\ldots\bar{D}^+\bar{\mathcal{W}}}{(\mathcal{W}\bar{\mathcal{W}})^{k+2}}(k+1)(-2q^-q^+)^k\\ &= \frac{1}{(4\pi)^2}\int d\zeta^{(-4)}du\frac{1}{16}\Big\{D^{+2}\ln\mathcal{W}\bar{D}^{+2}\ln\bar{\mathcal{W}}+\sum_{k=1}^{\infty}\frac{1}{k^2(k+1)}D^{+2}\frac{1}{\mathcal{W}^k}\bar{D}^{+2}\times\\ &\qquad\times\frac{1}{\bar{\mathcal{W}}^k}(-2q^-q^+)^k\Big\},\\ \Gamma_{F^4} &= \frac{1}{(4\pi)^2}\int d^{12}z\left\{\ln\mathcal{W}\ln\bar{\mathcal{W}}+\sum_{k=1}^{\infty}\frac{1}{k^2(k+1)}(\frac{-2q^{ai}q_{ai}}{\mathcal{W}\bar{\mathcal{W}}})^k\right\}.\end{aligned} \tag{6.4}$$

That exactly coincides with the earlier results[11] [24, 25, 27]:

$$\Gamma_{F^4} = \frac{1}{(4\pi)^2}\int d^{12}z\left\{\ln\mathcal{W}\ln\bar{\mathcal{W}}+\mathrm{Li}_2(X)+\ln(1-X)-\frac{1}{X}\ln(1-X)\right\},$$

where $X=\frac{-2q^{ai}q_{ai}}{\mathcal{W}\bar{\mathcal{W}}}$ and $\mathrm{Li}_2(X)$ is the Euler dilogarithm function.

[11] We use $\int d\zeta^{(-4)}(D^+)^4=\int d^{12}z$ and $\int du=1$ since in the central basis of the $\mathcal{N}=2$ harmonic superspace the hypermultiplet superfields are expressed on shell as $q^{\pm a}=q^{ia}u_i^{\pm}$ and $\mathcal{W}$ is harmonic independent.

Next-to-leading correction corresponds to F^8-term[12]. We have

$$\Gamma_{F^8} = \sum_{k=0}^{\infty} \frac{1}{2(4\pi)^2} \int d\zeta^{(-4)} du \frac{1}{16} \frac{D^+\mathcal{W} \ldots \bar{D}^+\bar{\mathcal{W}}}{(\mathcal{W}\bar{\mathcal{W}})^6} D^4\mathcal{W}^2 \bar{D}^4\bar{\mathcal{W}}^4 \times$$
$$\times \frac{1}{5!}(k+1)(k+2)(k+3)(k+4)(k+5) \left(\frac{-2q^- q^+}{\mathcal{W}\bar{\mathcal{W}}} \right)^k , \qquad (6.5)$$

Using the transformations

$$D^4\mathcal{W}^2 = 4D^+_\alpha D^{-\beta}\mathcal{W} D^{+\alpha} D^-_\beta \mathcal{W} ,$$

and

$$D^{+\alpha} D^+_\alpha (D^{+\delta}\mathcal{W} D^+_\delta \mathcal{W} D^{-\beta}\mathcal{W} D^-_\beta \mathcal{W}) = 2D^{+\delta}\mathcal{W} D^+_\delta \mathcal{W} D^+_\alpha D^{-\beta}\mathcal{W} D^{+\alpha} D^-_\beta \mathcal{W} ,$$

we get the chain of identities

$$\begin{aligned} \frac{1}{(\mathcal{W})^{k+6}} D^+\mathcal{W} D^+\mathcal{W} D^4\mathcal{W} &= 2D^+D^+ \left\{ \frac{D^+\mathcal{W} D^+\mathcal{W}}{\mathcal{W}^{k+4}} \frac{D^-\mathcal{W} D^-\mathcal{W}}{\mathcal{W}^2} \right\} \\ &= -2D^+D^+ \left\{ \frac{1}{(k+2)(k+3)} D^+D^+ \frac{1}{\mathcal{W}^{k+2}} D^-D^- \ln \mathcal{W} \right\} \\ &= -2D^+D^+ \left\{ \frac{1}{(k+2)(k+3)} \frac{1}{\mathcal{W}^{k+2}} D^4 \ln \mathcal{W} \right\} . \end{aligned} \qquad (6.6)$$

Similar transformations are done for the complex conjugate term. After restoration of the full measure $d^{12}z = d\zeta^{(-4)}(D^+)^4$ we have the factor

$$\frac{1}{(k+2)^2(k+3)^2} \frac{1}{\bar{\mathcal{W}}^2} D^4 \ln \mathcal{W} \frac{1}{\mathcal{W}^2} \bar{D}^4 \ln \bar{\mathcal{W}} \left(\frac{-2q^-_a q^{+a}}{\mathcal{W}\bar{\mathcal{W}}} \right)^k .$$

This allows to write Γ_{F^8} in the form [27]:

$$\begin{aligned} \Gamma_{F^8} &= \frac{1}{2(4\pi)^2 5!} \int d^{12}z \sum_{k=0}^{\infty} \frac{(k+1)(k+4)(k+5)}{(k+2)(k+3)} \Psi^2 \bar{\Psi}^2 \left(\frac{-2q^-_a q^{+a}}{\mathcal{W}\bar{\mathcal{W}}} \right)^k \qquad (6.7) \\ &= \frac{1}{2(4\pi)^2 5!} \int d^{12}z \Psi^2 \bar{\Psi}^2 \left\{ \frac{1}{(1-X)^2} + \frac{4}{(1-X)} + \frac{6X-4}{X^3} \ln(1-X) + 4\frac{X-1}{X^2} \right\} . \end{aligned}$$

It coincides with the result of $\mathcal{N}=1$ calculations derived on the base of above prescription [27].

The analogous consideration allows us in principle to get any term $\Gamma_{F^{2n}}$ of derivative expansion of the effective action (6.1). We pay attention that the integrals over analytic subspace are transformed to the integrals over full $\mathcal{N}=2$ superspace in each term of expansion.

[12] No F^6 quantum correction occurs in the one-loop effective action for $\mathcal{N}=4$ SYM [35, 5]

7. Summary

We have studied the problem of one-loop low-energy effective action in $\mathcal{N}=4$, SU(2) SYM theory. The theory is formulated in $\mathcal{N}=2$ harmonic superspace and possesses the manifest off-shell $\mathcal{N}=2$ supersymmetry and extra on-shell hidden $\mathcal{N}=2$ supersymmetry. We developed a new approach to derivations the effective action depending on all fields of $\mathcal{N}=4$ vector multiplet keeping the manifest $\mathcal{N}=2$ supersymmetry at all steps of calculations.

From $\mathcal{N}=2$ supersymmetric point of view, the effective action under consideration depends on fields of $\mathcal{N}=2$ vector multiplet and hypermultiplet. The theory is quantized in framework of the $\mathcal{N}=2$ background field method which allows us to obtain the effective action in manifestly gauge invariant form. We carried out the calculations of the effective action in low-energy approximation assuming the $\mathcal{N}=2$ superstrengths $\mathcal{W},\bar{\mathcal{W}}$ are on-shell and space-time independent and space-time independent hypermultiplet superfields q^a. The effective action is given by integral over analytic subspace of harmonic superspace of function depending on $\mathcal{W},\bar{\mathcal{W}}$, their spinor covariant derivatives and hypermultiplet superfields. This dependence is exactly found under the low-energy assumption. The effective action obtained is an extension of the results obtained in [24, 25] where only lowest powers of the spinor covariant derivatives of superfield strengths have been taken into account. We also check and justify the results of Ref. [27] where the one-loop effective action in hypermultiplet sector has been found in terms of $\mathcal{N}=1$ superfields using a special gauge fixing and some heuristic prescription about reconstruction of manifestly $\mathcal{N}=2$ supersymmetric form of effective action. In the given work this problem is solved automatically.

We have developed a general method to derive the one-loop, low-energy effective action of the theory under consideration completely in terms of $\mathcal{N}=2$ harmonic superspace. Starting point of our consideration is a series of covariant harmonic supergraphs with arbitrary number of external hypermultiplet legs. Each of such supergraphs is written as an integral over analytic subspace and all contributions are summed up. The result is given by expression (4.18). This expression is analyzed on the base of proper-time method and techniques of operator symbols (the basic aspects of this techniques in $\mathcal{N}=2$ harmonic superspace are developed in section 5) using the key relation (5.9) (this new relation is derived in appendix B). As a result we obtain the final expression (6.1) for the effective action.

At present there are at least two vast open problems concerning the hypermultiplet dependence of effective action in $\mathcal{N}=4$ SYM theory. The first one is a problem of effective action where the spinor covariant derivatives of hypermultiplet superfields are not vanishing. In this case, the low-energy effective action could be constructed in form of expansion in spinor derivative of superstrengths and hypermultiplet superfields. In particular, such an expansion would help to clarify an answer to the question whether or not the effective action is invariant under the hidden on-shell $\mathcal{N}=2$ supersymmetry like the classical action. The second one is a problem of hypermultiplet dependence of effective action at higher loops. We hope that the methods developed in this work will be useful for study both of the above problems.

JHEP09(2005)073

Acknowledgments

The authors are grateful to A.T. Banin and S.M. Kuzenko for useful discussions. I.L.B. is grateful to Trinity College, Cambridge for financial support. The work was supported in part by RFBR grant, project No 03-02-16193. The work of I.L.B. was also partially supported by INTAS grant, INTAS-03-51-6346, joint RFBR-DFG grant, project No 02-02-04002, DFG grant, project No 436 RUS 113/669, grant for LRSS, project No 1252.2003.2. The work of N.G.P. was supported in part by RFBR grant, project No 05-02-16211.

A. Useful representation of the Baker - Campbell - Hausdorff formula

The heat kernel associated with the operator $\mathcal{O}$ is given as matrix element of the operator $e^{\mathcal{O}}$. In many cases the operator $\mathcal{O}$ is a linear combination of basis operators which form some Lie (super)algebra. Then evaluation of the heat kernel is simplified drastically by using the Baker-Campbell-Hausdorf (BCH) fromula.[13] This formula states that the exponent of the sum of two noncommuting operators A and B can be presented as a series in powers of the commutator $[A, B]$.

We derive another representation of BCH-formula

$$e^{A+B} = e^{C_1+C_2+C_3+\cdots} e^{A}\,, \tag{A.1}$$

where the operators C_k are expressed through the commutators of the operators A and B. We show here that the operators C_k can be defined in such a way that each C_k has k-th power in operator B and infinite power in operator A. This representation of BCH-formula (A.1) can be useful in the case when the operators A and B are associated with some Lie (super)algebra which allows us to summarize the operator series $C_1+C_2+C_3+...$ in the explicit form.

We introduce in (A.1) the variables t as follows:

$$e^{A+tB} = e^{tC_1+t^2C_2+t^3C_3+\cdots} e^{A}\,. \tag{A.2}$$

Next, let us define the function

$$\mathcal{F} = e^{A+tB} e^{-A} = e^{\sum_{k=1}^{\infty} t^k C_k(A,B)} \tag{A.3}$$

and find the proper operators C_k in (A.2). It is obvious that at $t = 0$ the identity (A.2) takes place and at $t = 1$ ones get the initial relation (A.1). We calculate the logarithmic derivative of the function (A.2) with respect to t. On one hand, we have

$$\dot{\mathcal{F}}\mathcal{F}^{-1} = \int_0^1 d\tau e^{\tau A+\tau t B} B e^{-\tau A-\tau t B}\,. \tag{A.4}$$

On the other hand

$$\mathcal{F} = e^{tC_1+t^2C_2+\cdots}\,, \qquad \dot{\mathcal{F}}\mathcal{F}^{-1} = C_1 + 2tC_2 + \cdots\,. \tag{A.5}$$

[13]N.P. would like to thank L.L. Salcedo for enlightening and useful discussions on various aspects of the BCH formula

If we put $t = 0$, we find

$$C_1 = \int_0^1 d\tau e^{\tau A} B e^{-\tau A} = \int_0^1 d\tau \mathcal{B}(\tau)\,. \tag{A.6}$$

In order to obtain C_2 we have to calculate first derivative with respect t of the logarithmic derivation:

$$\begin{aligned}\frac{d}{dt}(\dot{\mathcal{F}}\mathcal{F}^{-1}) = &\int_0^1\int_0^1 d\tau' d\tau \{e^{\tau'\tau A+\tau'\tau t B}(\tau B)e^{-\tau'\tau A-\tau'\tau t B}\}\{e^{\tau A+\tau t B}Be^{-\tau A-\tau t B}\} + \\ &+ \int_0^1\int_0^1 d\tau' d\tau \{e^{\tau A+\tau t B}Be^{-\tau A-\tau t B}\}\{e^{\tau'\tau A+\tau'\tau t B}(-\tau B)e^{-\tau'\tau A-\tau'\tau t B}\}\,.\end{aligned} \tag{A.7}$$

At $t = 0$ we get the expression for C_2:

$$2C_2 = \int_0^1\int_0^1 d\tau d\tau' \cdot \tau \cdot [\mathcal{B}(\tau'\tau), \mathcal{B}(\tau)]\,. \tag{A.8}$$

Following the same way one can obtain all operators C_k. For example, to get the operator C_3 we have to calculate second derivation of above logarithmic derivative. It leads to

$$6C_3 + [C_1, C_2] = \int_0^1 d\tau'' d\tau' d\tau \cdot \tau^2 \{[\mathcal{B}(\tau'\tau), [\mathcal{B}(\tau''\tau), \mathcal{B}(\tau)]] - \tau'[\mathcal{B}(\tau), [\mathcal{B}(\tau''\tau'\tau), \mathcal{B}(\tau'\tau)]]\}\,, \tag{A.9}$$

where C_1 and C_2 have been obtained above. The operator C_4 is found from

$$\begin{aligned} &24C_4 + 6[C_1, C_3] + [C_1, [C_1, C_2]] = \\ &= \int_0^1 d\tau''' d\tau'' d\tau' d\tau \lceil \tau^2 \{\tau'\tau[[\mathcal{B}(\tau'''\tau'\tau), \mathcal{B}(\tau'\tau], [\mathcal{B}(\tau''\tau), \mathcal{B}(\tau)]] - \\ &\qquad - \tau''\tau[\mathcal{B}(\tau'\tau)[\mathcal{B}(\tau)[\mathcal{B}(\tau'''\tau''\tau), \mathcal{B}(\tau''\tau)] + \\ &\qquad + \tau[\mathcal{B}(\tau'\tau)[\mathcal{B}(\tau''\tau)[\mathcal{B}(\tau'''\tau), \mathcal{B}(\tau)]]]\} \times \\ &\quad \times \tau^2\tau'\{\tau''\tau'\tau[[[\mathcal{B}(\tau'''\tau''\tau'), \mathcal{B}(\tau''\tau'\tau)]\mathcal{B}(\tau'\tau)]\mathcal{B}(\tau)] + \\ &\qquad + \tau'\tau[[[\mathcal{B}(\tau''\tau'\tau)[\mathcal{B}(\tau'''\tau'\tau), \mathcal{B}(\tau'\tau)]\mathcal{B}(\tau)] + \\ &\qquad + \tau[[\mathcal{B}(\tau''\tau'\tau), \mathcal{B}(\tau'\tau)], [\mathcal{B}(\tau'''\tau), \mathcal{B}(\tau)]]\}\rfloor \end{aligned} \tag{A.10}$$

and etc. Following the same procedure we can obtain, in principle, all terms of the BCH series in (A.1). Note that when Lie (super)algebra under consideration has a central charge, this series terminates at some finite order and the representation (A.1) takes a simple form.

In the case under consideration, the problem is in rewriting the exponent (5.9) of sum of the operators satisfying the algebraic relations (5.5), (5.6) as a product of the exponents of individual operators. This problem is solved in two steps. First, we take A in (A.1) in the form $A^+D^- + \bar{A}^+\bar{D}^-$ and obtain C_1 as a linear combination of operators $\frac{1}{2}\mathcal{D}^{\alpha\dot{\alpha}}\mathcal{D}_{\alpha\dot{\alpha}}$, $f^{\alpha\dot{\alpha}}(A^\pm, \bar{A}^\pm, N, \bar{N})\mathcal{D}_{\alpha\dot{\alpha}}$ with some definite coefficients[14] plus some definite function of arguments $\mathcal{W}, \bar{\mathcal{W}}, A^\pm, \bar{A}^\pm, N, \bar{N}$ as the central element. This means, that all other operators

[14]indeed, e.g. $[A^+\mathcal{D}^-, \frac{1}{2}\mathcal{D}^{\alpha\dot{\alpha}}\mathcal{D}_{\alpha\dot{\alpha}}] = -A^{+\alpha}\bar{A}^{-\dot{\alpha}}\mathcal{D}_{\alpha\dot{\alpha}}$

$C_2, C_3, \dots$ will be proportional to the operator $\mathcal{D}_{\alpha\dot\alpha}$ with some coefficient functions. Therefore, the series $C_1 + C_2 + C_3 + \cdots$ is reduced to summing up these coefficients functions what can be done in explicit form. The result is $-s\frac{1}{2}\mathcal{D}^{\alpha\dot\alpha}\mathcal{D}_{\alpha\dot\alpha}$ plus $f^{\alpha\dot\alpha}(A^{\pm}, \bar A^{\pm}, N, \bar N)\mathcal{D}_{\alpha\dot\alpha}$ plus some central element. Second, we apply again the formula (A.1) to the expression $exp(-s\frac{1}{2}\mathcal{D}^{\alpha\dot\alpha}\mathcal{D}_{\alpha\dot\alpha} + f^{\alpha\dot\alpha}\mathcal{D}_{\alpha\dot\alpha})$ and take A as $-s\frac{1}{2}\mathcal{D}^{\alpha\dot\alpha}\mathcal{D}_{\alpha\dot\alpha}$. All operators C_k will be again proportional to the single operator $\mathcal{D}_{\alpha\dot\alpha}$ and series of coefficient functions can be summed up in explicit form. As a result we get the right had side of expression (5.9). All coefficient functions are given in appendix B. These coefficient functions can also be found from differential equations (5.10), (5.11). The results obtained from both BCH-formula and the above differential equations coincide.

B. Coefficient functions in decomposition of heat kernel

The linear differential equations (5.10) and (5.11) for $f_{\alpha\dot\alpha}(s)$ and $\Omega(s)$ can be solved exactly in the form (5.12), (5.13). Coefficients of the power expansion $f_{\alpha\dot\alpha}(s)$ over basis of Grassmann elements $A^+_\alpha, \bar A^+_{\dot\alpha}$ are given as follows:

$$\mathcal{N}^{\delta\dot\delta}_{\alpha\dot\alpha} = \int_0^s d\tau \left(\frac{e^{-\tau N} - e^{\tau\bar N}}{N+\bar N}\cdot e^{-s\bar N}\right)^{\delta\dot\delta}_{\alpha\dot\alpha} = -\frac{e^{-sF}-1}{NF} + \frac{e^{-s\bar N}-1}{N\bar N}, \tag{B.1}$$

$$\bar{\mathcal{N}}^{\dot\delta\delta}_{\dot\alpha\alpha} = \int_0^s d\tau \left(\frac{e^{-\tau\bar N} - e^{\tau N}}{N+\bar N}\cdot e^{-sN}\right)^{\dot\delta\delta}_{\dot\alpha\alpha} = -\frac{e^{-sF}-1}{\bar NF} + \frac{e^{-sN}-1}{N\bar N}. \tag{B.2}$$

Coefficients of the power expansion $\Omega(s)$ over basis of Grassmann elements $A^+_\alpha, \bar A^+_{\dot\alpha}$ are given as follow:

$$\Omega^-_\alpha = -\bar{\mathcal{W}}\left\{\frac{e^{-sN}+sN-1}{N^2}\right\}^{\beta}_{\alpha} A^-_\beta, \tag{B.3}$$

$$\bar\Omega^-_{\dot\alpha} = -\mathcal{W}\bar A^-_{\dot\beta}\left\{\frac{e^{-s\bar N}+s\bar N-1}{\bar N^2}\right\}^{\dot\beta}_{\dot\alpha}, \tag{B.4}$$

$$\Psi^{(-2)} = \frac{1}{8}(\bar A^-)^2 \mathrm{tr}\sum_{n=0}^{\infty}\sum_{p=1}^{n}\frac{s^{n+2}}{(n+2)!}C^p_n(-F)^{n-p}\left\{N^{p-1} - (-1)^n\bar N^{p-1}\right\} \tag{B.5}$$

$$= (\bar A^-)^2\left\{\frac{s^3}{6} + \frac{s^5}{5!}(N^2+\bar N^2) + \cdots\right\}, \tag{B.6}$$

$$\bar\Psi^{(-2)} = \frac{1}{8}(A^-)^2 \mathrm{tr}\sum_{n=0}^{\infty}\sum_{p=1}^{n}\frac{s^{n+2}}{(n+2)!}C^p_n(-F)^{n-p}\{\bar N^{p-1} - (-1)^n N^{p-1}\} \tag{B.7}$$

$$= (A^-)^2\left\{\frac{s^3}{6} + \frac{s^5}{5!}(N^2+\bar N^2) + \cdots\right\}, \tag{B.8}$$

$$\Psi^{\dot\alpha(-2)}_\alpha = \Psi^{\dot\alpha\delta}_{\alpha\dot\delta}A^-_\delta\bar A^{-\dot\delta}, \tag{B.9}$$

$$\Psi^{\dot\alpha\delta}_{\alpha\dot\delta} = \frac{1}{N\bar N(N-\bar N)} + \frac{1}{N\bar N}\left\{\frac{e^{-sN}}{\bar N} - \frac{e^{s\bar N}}{N}\right\} +$$

$$+\frac{N^2+\bar{N}^2}{2N^2\bar{N}^2(\bar{N}-N)}e^{s(\bar{N}-N)}+\frac{\bar{N}^2e^{sF}-N^2e^{-sF}}{2N^2\bar{N}^2(N+\bar{N})} \tag{B.10}$$

$$=\frac{s^3}{3}+\frac{s^4}{8}(\bar{N}-N)+\frac{7s^5}{5!}(N^2+\bar{N}^2)+\cdots. \tag{B.11}$$

Coefficients of the derivative expansion $\exp\{-s(A^+\mathcal{D}^-+\bar{A}^+\bar{\mathcal{D}}^-)\}$ defined in (5.14) are given as follows:

$$a^{+\alpha} = A^{+\beta}\left(\frac{e^{-sN}-1}{N}\right)_\beta^{\ \alpha}, \qquad \bar{a}^{+\dot{\alpha}} = \bar{A}^{+\dot{\beta}}\left(\frac{e^{-s\bar{N}}-1}{\bar{N}}\right)_{\dot{\beta}}^{\ \dot{\alpha}}, \tag{B.12}$$

$$f^{+2} = -\frac{1}{4}(A^+)^2\mathrm{tr}\left(\frac{\cosh(sN)-1}{N^2}\right), \qquad \bar{f}^{+2} = -\frac{1}{4}(\bar{A}^+)^2\mathrm{tr}\left(\frac{\cosh(s\bar{N})-1}{\bar{N}^2}\right), \tag{B.13}$$

$$f^{+2\dot{\alpha}\alpha} = -A^{+\beta}\bar{A}^{+\dot{\beta}}\left(\frac{e^{-sN}-1}{N}\right)_\beta^{\ \alpha}\left(\frac{e^{-s\bar{N}}-1}{\bar{N}}\right)_{\dot{\beta}}^{\ \dot{\alpha}}, \tag{B.14}$$

$$\bar{\Xi}^{+3\dot{\alpha}} = -\frac{1}{4}(A^+)^2\bar{A}^{+\dot{\beta}}\left(\frac{e^{-s\bar{N}}-1}{\bar{N}}\right)_{\dot{\beta}}^{\ \dot{\alpha}}\mathrm{tr}\left(\frac{\cosh(sN)-1}{N^2}\right), \tag{B.15}$$

$$\Xi^{+3\alpha} = -\frac{1}{4}(\bar{A}^+)^2A^{+\beta}\left(\frac{e^{-sN}-1}{N}\right)_\beta^{\ \alpha}\mathrm{tr}\left(\frac{\cosh(s\bar{N}-1)}{\bar{N}^2}\right), \tag{B.16}$$

$$\Omega^{+4} = -\frac{1}{16}(A^+)^2(\bar{A}^+)^2\mathrm{tr}\left(\frac{\cosh(sN)-1}{N^2}\right)\mathrm{tr}\left(\frac{\cosh(s\bar{N})-1}{\bar{N}^2}\right). \tag{B.17}$$

References

[1] I.W. Taylor, *Lectures on D-branes, gauge theory and M(atrices)*, `hep-th/9801182`.

[2] E. D'Hoker and D.Z. Freedman, *Supersymmetric gauge theories and the AdS/CFT correspondence*, `hep-th/0201253`.

[3] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [`hep-th/9711200`];
S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B 428** (1998) 105 [`hep-th/9802109`];
E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [`hep-th/9802150`];
O. Aharony, S.S. Gubser, J.M. Maldacena, H. Ooguri and Y. Oz, *Large-N field theories, string theory and gravity*, *Phys. Rept.* **323** (2000) 183 [`hep-th/9905111`].

[4] I. Chepelev and A.A. Tseytlin, *Long-distance interactions of branes: correspondence between supergravity and super Yang-Mills descriptions*, *Nucl. Phys.* **B 515** (1998) 73 [`hep-th/9709087`];
A.A. Tseytlin, *Born-Infeld action, supersymmetry and string theory*, `hep-th/9908105`.

[5] I.L. Buchbinder, S.M. Kuzenko and A.A. Tseytlin, *On low-energy effective actions in N = 2,4 superconformal theories in four dimensions*, *Phys. Rev.* **D 62** (2000) 045001 [`hep-th/9911221`].

[6] I.L. Buchbinder, A.Y. Petrov and A.A. Tseytlin, *Two-loop N = 4 super Yang-Mills effective action and interaction between D3-branes*, *Nucl. Phys.* **B 621** (2002) 179 [`hep-th/0110173`].

JHEP09(2005)073

[7] A. Galperin, E. Ivanov, S. Kalitsyn, V. Ogievetsky and E. Sokatchev, *$N = 3$ supersymmetric gauge theory*, *Phys. Lett.* **B 151** (1985) 215; *Unconstrained off-shell $N = 3$ supersymmetric Yang-Mills theory*, *Class. and Quant. Grav.* **2** (1985) 155;
A.S. Galperin, E.A. Ivanov and V.I. Ogievetsky, *Superspaces for $N = 3$ supersymmetry*, *Sov. J. Nucl. Phys.* **46** (1987) 543.

[8] A.S. Galperin, E.A. Ivanov, V.I. Ogievetsky and E.S. Sokatchev, *Harmonic superspace*, Cambridge Univ. Press, 2001, p. 306.

[9] F. Delduc and J. McCabe, *The quantization of $N = 3$ super Yang-Mills off-shell in harmonic superspace*, *Class. and Quant. Grav.* **6** (1989) 233.

[10] I.L. Buchbinder, E.A. Ivanov, I.B. Samsonov and B.M. Zupnik, *Scale invariant low-energy effective action in $N = 3$ SYM theory*, *Nucl. Phys.* **B 689** (2004) 91 [hep-th/0403053].

[11] E.A. Ivanov and B.M. Zupnik, *$N = 3$ supersymmetric Born-Infeld theory*, *Nucl. Phys.* **B 618** (2001) 3 [hep-th/0110074];
E. Ivanov, *Towards higher-n superextensions of Born-Infeld theory*, *Russ. Phys. J.* **45** (2002) 695–708 [hep-th/0202201].

[12] A. Galperin, E. Ivanov, S. Kalitsyn, V. Ogievetsky and E. Sokatchev, *Unconstrained $N = 2$ matter, Yang-Mills and supergravity theories in harmonic superspace*, *Class. and Quant. Grav.* **1** (1984) 469; *Harmonic supergraphs. Green functions*, *Class. and Quant. Grav.* **2** (1985) 601; *Harmonic supergraphs. Feynman rules and examples*, *Class. and Quant. Grav.* **2** (1985) 617.

[13] N. Ohta and H. Yamaguchi, *Superfield perturbation theory in harmonic superspace*, *Phys. Rev.* **D 32** (1985) 1954.

[14] I.L. Buchbinder, E.I. Buchbinder, S.M. Kuzenko and B.A. Ovrut, *The background field method for $N = 2$ super Yang-Mills theories in harmonic superspace*, *Phys. Lett.* **B 417** (1998) 61 [hep-th/9704214];
I. Buchbinder, S. Kuzenko and B.A. Ovrut, *Covariant harmonic supergraphity for $N = 2$ super Yang-Mills theories*, hep-th/9810040.

[15] E.I. Buchbinder, I.L. Buchbinder, E.A. Ivanov, S.M. Kuzenko, B.A. Ovrut, *Low-energy effective action in $N = 2$ supersymmetric field theories*, *Physics of Particles and Nuclei*, **32** (2001) 641.

[16] S.M. Kuzenko and I.N. McArthur, *On the background field method beyond one loop: a manifestly covariant derivative expansion in super Yang-Mills theories*, *JHEP* **05** (2003) 015 [hep-th/0302205]; *Low-energy dynamics in $N = 2$ super QED: two-loop approximation*, *JHEP* **10** (2003) 029 [hep-th/0308136]; *On the two-loop four-derivative quantum corrections in 4D $N = 2$ superconformal field theories*, *Nucl. Phys.* **B 683** (2004) 3 [hep-th/0310025]; *Relaxed super self-duality and effective action*, *Phys. Lett.* **B 591** (2004) 304 [hep-th/0403082]; *Relaxed super self-duality and $N = 4$ SYM at two loops*, *Nucl. Phys.* **B 697** (2004) 89 [hep-th/0403240].

[17] P.J. Heslop and P.S. Howe, *Aspects of $N = 4$ SYM*, *JHEP* **01** (2004) 058 [hep-th/0307210];
J.M. Drummond, P.J. Heslop, P.S. Howe and S.F. Kerstan, *Integral invariants in $N = 4$ SYM and the effective action for coincident D-branes*, *JHEP* **08** (2003) 016 [hep-th/0305202];
P.C. Argyres, A.M. Awad, G.A. Braun and F.P. Esposito, *Higher-derivative terms in $N = 2$ supersymmetric effective actions*, *JHEP* **07** (2003) 060 [hep-th/0306118].

[18] P.C. Argyres, M.R. Plesser and N. Seiberg, *The moduli space of $N = 2$ SUSY QCD and duality in $N = 1$ SUSY QCD*, *Nucl. Phys.* **B 471** (1996) 159 [hep-th/9603042].

[19] M. Dine and N. Seiberg, *Comments on higher derivative operators in some SUSY field theories*, *Phys. Lett.* **B 409** (1997) 239 [hep-th/9705057];
D.A. Lowe and R. von Unge, *Constraints on higher derivative operators in maximally supersymmetric gauge theory*, *JHEP* **11** (1998) 014 [hep-th/9811017];
F. Gonzalez-Rey and M. Roček, *Nonholomorphic $N = 2$ terms in $N = 4$ SYM: 1-loop calculation in $N = 2$ superspace*, *Phys. Lett.* **B 434** (1998) 303 [hep-th/9804010];
V. Periwal and R. von Unge, *Accelerating D-branes*, *Phys. Lett.* **B 430** (1998) 71 [hep-th/9801121].

[20] I.L. Buchbinder and S.M. Kuzenko, *Comments on the background field method in harmonic superspace: non-holomorphic corrections in $N = 4$ SYM*, *Mod. Phys. Lett.* **A 13** (1998) 1623 [hep-th/9804168];
E.I. Buchbinder, I.L. Buchbinder and S.M. Kuzenko, *Non-holomorphic effective potential in $N = 4$ SU(N) SYM*, *Phys. Lett.* **B 446** (1999) 216 [hep-th/9810239].

[21] I.L. Buchbinder, S.M. Kuzenko, B.A. Ovrut, *On the $D = 4$, $N = 2$ non-renormalization theorem*, *Phys. Lett.* **B 433** (1988) 335 [hep-th/9710142];
I.L. Buchbinder, A.Yu. Petrov, *$N = 4$ super Yang-Mills low-energy effective action at three and four loops*, *Phys. Lett.* **B 482** (2000) 429 [hep-th/0003265].

[22] B. de Wit, M.T. Grisaru and M. Roček, *Nonholomorphic corrections to the one-loop $N = 2$ super Yang-Mills action*, *Phys. Lett.* **B 374** (1996) 297 [hep-th/9601115];
A. De Giovanni, M.T. Grisaru, M. Roček, R. von Unge and D. Zanon, *The $N = 2$ super Yang-Mills low-energy effective action at two loops*, *Phys. Lett.* **B 409** (1997) 251 [hep-th/9706013];
A. Yung, *Higher derivative terms in the effective action of $N = 2$ SUSY QCD from instantons*, *Nucl. Phys.* **B 512** (1998) 79 [hep-th/9705181];
D. Bellisai, F. Fucito, M. Matone and G. Travaglini, *Non-holomorphic terms in $N = 2$ SUSY wilsonian actions and RG equation*, *Phys. Rev.* **D 56** (1997) 5218 [hep-th/9706099];
N. Dorey, V.V. Khoze, M.P. Mattis, M.J. Slater and W.A. Weir, *Instantons, higher-derivative terms and nonrenormalization theorems in supersymmetric gauge theories*, *Phys. Lett.* **B 408** (1997) 213 [hep-th/9706007].

[23] K. Becker and M. Becker, *A two-loop test of m(atrix) theory*, *Nucl. Phys.* **B 506** (1997) 48 [hep-th/9705091];
K. Becker, *Testing m(atrix) theory at two loops*, *Nucl. Phys.* **68** *(Proc. Suppl.)* (1998) 165 [hep-th/9709038];
S. Sethi, *Structure in supersymmetric Yang-Mills theory*, *JHEP* **10** (2004) 001 [hep-th/0404056].

[24] I.L. Buchbinder and E.A. Ivanov, *Complete $N = 4$ structure of low-energy effective action in $N = 4$ super Yang-Mills theories*, *Phys. Lett.* **B 524** (2002) 208 [hep-th/0111062].

[25] I.L. Buchbinder, E.A. Ivanov and A.Y. Petrov, *Complete low-energy effective action in $N = 4$ SYM: a direct $N = 2$ supergraph calculation*, *Nucl. Phys.* **B 653** (2003) 64 [hep-th/0210241].

[26] I.N. McArthur and T.D. Gargett, *A 'gaussian' approach to computing supersymmetric effective actions*, *Nucl. Phys.* **B 497** (1997) 525 [hep-th/9705200].

[27] A.T. Banin, I.L. Buchbinder and N.G. Pletnev, *One-loop effective action for $N = 4$ SYM theory in the hypermultiplet sector: leading low-energy approximation and beyond*, *Phys. Rev.* **D 68** (2003) 065024 [hep-th/0304046];
A.T. Banin and N.G. Pletnev, *On the construction of $N = 4$ SYM effective action beyond leading low-energy approximation*, hep-th/0401006; *Low-energy next-to-leading contributions to the effective action in $N = 4$ SYM theory*, hep-th/0401007.

[28] N.G. Pletnev and A.T. Banin, *Covariant technique of derivative expansion of one-loop effective action*, *Phys. Rev.* **D 60** (1999) 105017 [hep-th/9811031];
A.T. Banin, I.L. Buchbinder and N.G. Pletnev, *Low-energy effective action of $N = 2$ gauge multiplet induced by hypermultiplet matter*, *Nucl. Phys.* **B 598** (2001) 371 [hep-th/0008167];
On low-energy effective action in $N = 2$ super Yang-Mills theories on non-abelian background, *Phys. Rev.* **D 66** (2002) 045021 [hep-th/0205034]; *Chiral effective potential in $N = 1/2$ non-commutative Wess- Zumino model*, *JHEP* **07** (2004) 011 [hep-th/0405063].

[29] A. Galperin, N.A. Ky and E. Sokatchev, *Coinciding harmonic singularities in harmonic supergraphs*, *Mod. Phys. Lett.* **A 2** (1987) 33.

[30] P.S. Howe, K.S. Stelle and P.K. Townsend, *Miraculous ultraviolet cancellations in supersymmetry made manifest*, *Nucl. Phys.* **B 236** (1984) 125.

[31] S.M. Kuzenko, *Exact propagators in harmonic superspace*, *Phys. Lett.* **B 600** (2004) 163 [hep-th/0407242].

[32] S.J. Gates, M.T. Grisaru, M. Roček and W. Siegel, *Superspace, or one thousand and one lessons in supersymmetry*, *Front. Phys.* **58** (1983) 1–548 [hep-th/0108200].

[33] I.L. Buchbinder, S.M. Kuzenko, *Ideas and methods of supersymmetry and supergravity or a walk through superspace*, IOP, Bristol, UK, 1998.

[34] S.M. Kuzenko and I.N. McArthur, *Effective action of $N = 4$ super Yang-Mills: $N = 2$ superspace approach*, *Phys. Lett.* **B 506** (2001) 140 [hep-th/0101127]; *Hypermultiplet effective action: $N = 2$ superspace approach*, *Phys. Lett.* **B 513** (2001) 213 [hep-th/0105121].

[35] E.S. Fradkin and A.A. Tseytlin, *Quantization and dimensional reduction: one loop results for super Yang-Mills and supergravities in $d > 4$*, *Phys. Lett.* **B 123** (1983) 231; *Quantum properties of higher dimensional and dimensionally reduced supersymmetric theories*, *Nucl. Phys.* **B 277** (1983) 252.